## Studies in Fuzziness and Soft Computing

## Zeshui Xu

## Hesitant Fuzzy Sets

 Theory
# Studies in Fuzziness and Soft Computing 

Volume 314

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## Hesitant Fuzzy Sets Theory

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ISSN 1434-9922 ISSN 1860-0808 (electronic)
ISBN 978-3-319-04710-2 ISBN 978-3-319-04711-9 (eBook)
DOI 10.1007/978-3-319-04711-9
Springer Cham Heidelberg New York Dordrecht London
Library of Congress Control Number: 2013958274
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## Preface

When people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. For example, two decision makers discuss the membership degree of an element to a set, and one wants to assign 0.6 but the other 0.8 . Accordingly, the difficulty of establishing a common membership degree is not because we have a margin of error, or some possibility distribution values, but because we have a set of possible values. To deal with such cases, Torra and Narukawa (2009) introduced the concept of hesitant fuzzy set (HFS). The HFS, as one of the extensions of Zadeh (1965)'s fuzzy set, allows the membership degree that an element to a set presented by several possible values, and can express the hesitant information more comprehensively than other extensions of fuzzy set. In 2011, Xu and Xia defined the concept of hesitant fuzzy element (HFE), which can be considered as the basic unit of a HFS, and is also a simple and effective tool used to express the decision makers' hesitant preferences in the process of decision making. Since then, our research group has done lots of research work on aggregation, distance, similarity and correlation measures, clustering analysis, and decision making with hesitant fuzzy information.

In this book, we give a thorough and systematic introduction to the main research results in hesitant fuzzy theory, which include the hesitant fuzzy aggregation techniques, the hesitant fuzzy preference relations, the hesitant fuzzy measures, the hesitant fuzzy clustering algorithms, and the hesitant fuzzy multiattribute decision making methods, etc. We organize this book into four chapters that deal with four different but related issues, which are listed below:

Chapter 1 introduces a series of hesitant fuzzy aggregation operators. We first introduce the hesitant fuzzy elements (HFEs), give their comparison methods, basic operational laws, and their desirable properties. Based on these operations, we develop lots of operators for aggregating HFEs, such as the hesitant fuzzy weighted aggregation operators, the generalized hesitant fuzzy weighted aggregation operators, the hesitant fuzzy ordered weighted aggregation operators, the generalized hesitant fuzzy ordered weighted aggregation operators, the hesitant fuzzy hybrid aggregation operators, the generalized hesitant fuzzy hybrid aggregation operators, the hesitant fuzzy Bonferroni means, the hesitant fuzzy aggregation operators based on quasi-arithmetic means and the induced idea, the hesitant fuzzy aggregation operators based on t-norms and t-conorms, the hesitant multiplicative aggregation operators, discuss their relations in detail, and apply them to the enterprise's development planning of strategy initiatives, site
selection, the supplier selection in a supply chain, safety evaluation of work systems, etc.

Chapter 2 mainly investigates the distance, similarity, correlation, entropy measures and clustering algorithms for hesitant fuzzy information. We first introduce a series of distance measures for HFSs, based on which the corresponding similarity measures are given. Then we investigate the distance and correlation measures for HFEs, and discuss their properties in detail. We introduce the concepts of entropy and crossentropy for hesitant fuzzy information, and analyze the relationships among the proposed entropy, cross-entropy, and similarity measures. We also introduce some correlation coefficient formulas and use them to calculate the degrees of correlation among HFSs aiming at clustering different objects. Moreover, we give the hesitant fuzzy agglomerative hierarchical clustering algorithm, the hierarchical hesitant fuzzy K-means clustering algorithm which takes the results of hierarchical clustering as the initial input, and also introduce a minimal spanning tree algorithm-based clustering technique to make clustering analysis of HFSs via some hesitant fuzzy distances. The applications of the algorithms in energy policy evaluation, medical diagnosis, supplier selection of manufacturing enterprise, software evaluation and classification, and tourism resources assessment, etc., are demonstrated.

Chapter 3 focuses on group decision making with hesitant preference relations. We introduce the concepts of hesitant fuzzy preference relation and multiplicative preference relation, by using them we give two group decision making approaches. Based on the multiplicative consistency and the acceptable multiplicative consistency, we establish two algorithms to improve the inconsistency level of a hesitant fuzzy preference relation, and investigate the consensus of group decision making based on hesitant fuzzy preference relations. We introduce two regression methods that transform hesitant fuzzy preference relations into fuzzy preference relations, which depend on the additive transitivity and the weak consistency respectively. Based on two principles (i.e., $\alpha$ normalization and $\beta$-normalization), we develop a hesitant goal programming model to derive priorities from hesitant fuzzy preference relations and some consistency measures of hesitant fuzzy preference relations. Additionally, we introduce a hesitant fuzzy programming method to derive priorities from a hesitant multiplicative preference relation in AHP-hesitant group decision making.

Chapter 4 is devoted to the multi-attribute decision making models with hesitant fuzzy information. Based on the TOPSIS and the maximizing deviation method, we give an approach for solving the multi-attribute decision making problems, in which the evaluation information provided by the decision maker is expressed in HFEs and the information about attribute weights is incomplete. By using the concepts of hesitant fuzzy concordance and hesitant fuzzy discordance which are based on the given scores and the deviation degrees, we introduce a hesitant fuzzy ELECTRE I method and apply it to solve the multi-attribute decision making problem with hesitant fuzzy information. With the incomplete weight information, we define the satisfaction degree of an alternative, based on which several optimization models are derived to determine the weights of attributes, and then develop an interactive method based on some optimization models for the multi-attribute decision making problems under hesitant fuzzy
environments. Moreover, all the given methods are illustrated and used in some practical applications.

This book can be used as a reference for researchers and practitioners working in the fields of fuzzy mathematics, operations research, information science, management science and engineering, etc. It can also be used as a textbook for postgraduate and senior-year undergraduate students.

This work was supported by the National Natural Science Foundation of China under Grant 61273209.

Special thanks to Dr. Meimei Xia for providing lots of useful material.

December 2013
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Chengdu

## Contents

1 Hesitant Fuzzy Aggregation Operators and Their Applications ..... 1
1.1 Hesitant Fuzzy Elements ..... 2
1.1.1 Comparison Methods ..... 2
1.1.2 Basic Operations and Relations ..... 6
1.2 Hesitant Fuzzy Aggregation Operators ..... 30
1.3 Hesitant Fuzzy Bonferroni Means ..... 66
1.4 Hesitant Fuzzy Geometric Bonferroni Means ..... 80
1.5 Hesitant Fuzzy Aggregation Operators Based on Quasi-Arithmetic Means and Induced Idea ..... 99
1.6 Generalized Hesitant Fuzzy Aggregation. ..... 132
1.7 Hesitant Multiplicative Aggregation ..... 145
2 Distance, Similarity, Correlation, Entropy Measures and Clustering Algorithms for Hesitant Fuzzy Information ..... 165
2.1 Distance and Similarity Measures for HFSs ..... 169
2.2 Distance and Correlation Measures for HFEs ..... 183
2.3 Hesitant Fuzzy Entropy and Cross-Entropy and Their Use in MADM ..... 197
2.4 Correlation Coefficients of HFSs and Their Applications to Clustering Analysis ..... 218
2.5 Hesitant Fuzzy Agglomerative Hierarchical Clustering Algorithms ..... 235
2.6 Hierarchical Hesitant Fuzzy K-means Clustering Algorithm ..... 249
2.7 MST Clustering Algorithm for HFSs ..... 262
2.7.1 Graph and Minimal Spanning Trees ..... 262
2.7.2 HFMST Clustering Algorithm ..... 263
2.7.3 Numerical Examples ..... 265
3 Hesitant Preference Relations ..... 281
3.1 Hesitant Fuzzy Preference Relations in Group Decision Making ..... 283
3.2 Hesitant Multiplicative Preference Relations ..... 288
3.3 Transitivity and Multiplicative Consistency on Hesitant Fuzzy Preference Relation ..... 293
3.3.1 Some Properties of Hesitant Fuzzy Preference Relation ..... 294
3.3.2 Iterative Algorithm for Improving Consistency of Hesitant Fuzzy Preference Relation ..... 298
3.3.3 Approach to Group Decision Making Based on Multiplicative Consensus of Hesitant Fuzzy Preference Relations ..... 302
3.4 Regression Methods for Hesitant Fuzzy Preference Relations ..... 316
3.4.1 Regression Method of Hesitant Fuzzy Preference Relations Based on Additive Transitivity ..... 317
3.4.2 Regression Method of Hesitant Fuzzy Preference Relations Based on Weak Consistency ..... 324
3.5 Deriving a Ranking from Hesitant Fuzzy Preference Relations under Group Decision Making ..... 332
3.5.1 Deriving Priorities from Hesitant Fuzzy Preference Relations with $\alpha$-Normalization ..... 332
3.5.2 Deriving Priorities from Hesitant Fuzzy Preference Relations with $\beta$-Normalization ..... 336
3.6 Deriving Priorities in AHP-Hesitant Group Decision Making ..... 354
3.6.1 Description of the Prioritization Method ..... 357
3.6.2 Hesitant Fuzzy Programming Method ..... 359
3.6.3 Numerical Examples ..... 366
4 Hesitant Fuzzy MADM Models ..... 379
4.1 Hesitant Fuzzy MADM Based on TOPSIS with Incomplete Weight Information ..... 381
4.2 ELECTRE I Method for Hesitant Fuzzy MADM ..... 395
4.3 Interactive Decision Making Method under Hesitant Fuzzy Environment with Incomplete Weight Information ..... 432
4.3.1 Satisfaction Degree Based Models for MADM with Incomplete Weight Information ..... 432
4.3.2 Interactive Method for MADM under Hesitant Fuzzy Environment with Incomplete Weight Information ..... 442
References ..... 449

## Chapter 1 <br> Hesitant Fuzzy Aggregation Operators and Their Applications

Since fuzzy set (Zadeh 1965) was introduced, several extensions have been developed, such as intutionistic fuzzy set (Atanassov 1986), type-2 fuzzy set (Dubois and Prade 1980; Miyamoto 2005), type- $n$ fuzzy set (Dubois and Prade 1980), fuzzy multiset (Yager 1986; Miyamoto 2000) and hesitant fuzzy set (Torra and Narukawa 2009; Torra 2010; Zhu et al. 2012a). Intuitionistic fuzzy set has three main parts: membership function, non-membership function and hesitancy function. Type-2 fuzzy set allows the membership of a given element as a fuzzy set. Type- $n$ fuzzy set generalizes type- 2 fuzzy set permitting the membership to be type- $n-1$ fuzzy set. In fuzzy multiset, the elements can be repeated more than once. Hesitant fuzzy set (HFS) permits the membership having a set of possible values. Torra and Narukawa (2009) and Torra (2010) discussed the relationship between the HFS and other three kinds of fuzzy sets, and showed that the envelope of HFS is an intuitionistic fuzzy set. He also proved that the operations he proposed are consistent with the ones of intitionistic fuzzy sets when applied to the envelopes of hesitant fuzzy sets.

HFSs can be applied in many decision making problems. To get the optimal alternative in a decision making problem with multiple attributes and multiple persons, there are usually two ways (Xia and Xu 2011a): (1) Aggregate the decision makers (DMs)' opinions under each attribute for alternatives, and then aggregate the collective values of attributes for each alternative; (2) Aggregate the attribute values given by the DMs for each alternative, and then aggregate the DMs' opinions for each alternative. For example, for a decision making problem with four attributes $x_{j}(j=1,2,3,4)$, five DMs $D_{k}(k=1,2,3,4,5)$ are required to give the attribute values of three alternatives $A_{i}(i=1,2,3)$. If we have known that $D_{1}$ is familiar with $x_{1}, D_{2}$ with $x_{2}, D_{3}$ with $x_{3}, D_{4}$ and $D_{5}$ with $x_{4}$, then it is better to let the DM evaluate the attribute that he/she is familiar to, so as to make the decision information more reasonable. However, in some practical problems, anonymity is required in order to protect the DMs' privacy or avoid influencing each other, for instance, the presidential election or the blind peer review of thesis, in which we don't know which attributes that the

DMs are respectively familiar with, and thus, leading us to consider all the situations in order to get more reasonable decision results. But the existing methods only consider the minor situations that each DM is good at evaluating all the attributes, which hardly happen. The HFS is very useful in avoiding such issues in which each alternative can be described in a HFS defined in terms of the opinions of the DMs (Torra and Narukawa 2009), and each attribute under the alternative can be depicted by a hesitant fuzzy element (HFE) (Xu and Xia 2011a). Then the aggregation techniques should be given to fuse the HFEs for each alternative under the attributes. In the chapter, we should give a detailed introduction to the existing aggregation operators for hesitant fuzzy information, and their applications to multi-attribute decision making (MADM).

### 1.1 Hesitant Fuzzy Elements

### 1.1.1 Comparison Methods

When people make a decision, they are usually hesitant and irresolute for one thing or another, which makes it difficult to reach a final agreement. The difficulty of establishing a common membership degree is not because we have one possible value (fuzzy set), or a margin of error (intuitionistic fuzzy set, interval-valued fuzzy set (Zadeh 1975), but because we have a set of possible values. In most cases, to get a more reasonable decision result, a decision organization, which contains a lot of DMs (or experts), is authorized to provide the preference information about a set of alternatives. Usually, the decision organization is not very sure about a value, but has hesitancy between several possible values, when it estimates the degrees that an alternative should satisfy an attribute (Xu and Xia 2012a). For example, some DMs in the decision organization provide 0.3 , some provide 0.5 , and the others provide 0.6 , and when these three parts cannot persuade each other, the degrees that the alternative should satisfy the criterion can be represented by a hesitant fuzzy element (HFE) $\{0.3,0.5,0.6\}$ (Xu and Xia 2012a). It is noted that the $\operatorname{HFE}\{0.3,0.5,0.6\}$, which can be considered as the basic unit of a hesitant fuzzy set (Torra and Narukawa 2009), can describe the above situation more comprehensively than the crisp number 0.3 (or 0.6 ), or the interval-valued fuzzy number $[0.3,0.6]$, or the intuitionistic fuzzy number (IFN) ( $0.3,0.4$ ) (Xu and Yager 2006; Xu 2007a), because the degrees that the alternative should satisfy the attribute are not the convex of 0.3 and 0.6 , or the interval between 0.3 and 0.6 , but just three possible values $0.3,0.5$ and 0.6. To deal with such cases, Torra and Narukawa (2009), and Torra (2010) proposed another generation of fuzzy set:

Definition 1.1 (Torra and Narukawa 2009; Torra 2010). Let $X$ be a fixed set, a hesitant fuzzy set (HFS) on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$.

To be easily understood, Xia and Xu (2011a) expressed the HFS by a mathematical symbol:

$$
\begin{equation*}
A=\left\{<x, h_{A}(x)>\mid x \in X\right\} \tag{1.1}
\end{equation*}
$$

where $h_{A}(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $A . \mathrm{Xu}$ and Xia (2011b) called $h=h_{A}(x)$ a hesitant fuzzy element (HFE) and $\Theta$ the set of all HFEs.

Example 1.1 (Chen et al. 2013a). Let $X=\left\{x_{1}, x_{2}, x_{3}\right\}$ be a fixed set, $h_{A}\left(x_{1}\right)=\{0.2,0.4,0.5\}, h_{A}\left(x_{2}\right)=\{0.3,0.4\}$ and $h_{A}\left(x_{3}\right)=\{0.3,0.2,0.5,0.6\}$ be the HFEs of $x_{i}(i=1,2,3)$ to a set $A$, respectively. Then $A$ can be considered as a HFS:

$$
A=\left\{<x_{1},\{0.2,0.4,0.5\}>,<x_{2},\{0.3,0.4\}>,<x_{3},\{0.3,0.2,0.5,0.6\}>\right\}
$$

Torra (2010) gave some special HFEs for $x$ in $X$ :
(1) Empty set: $h=\{0\}$, we denotes it as $O^{*}$ for simplification.
(2) Full set: $h=\{1\}$, denoted as $I^{*}$.
(3) Complete ignorance (all are possible): $h=[0,1] \triangleq U^{*}$.
(4) Nonsense set: $h=\varnothing^{*}$.

Liao and Xu (2013a) made some deep clarifications on these special HFEs from the viewpoint of the definition of HFS and also from the practical decision making processes: As presented in Definition 1.1 and Eq.(1.1), the HFS on the fixed set $X$ is in terms of a function $h$ that when applied to $X$ returns a subset of $[0,1]$. Hence, if $h$ returns no value, it is adequate for us to assert that $h$ is a nonsense set. Analogously, if it returns the set $[0,1]$, which means all values between 0 and 1 are possible, we call it complete ignorance. Particularly, if it returns only one value $\gamma \in[0,1]$, this certainly makes sense because the single value $\gamma \in[0,1]$ can also be seen as a subset of $[0,1]$, i.e., we can take $\gamma$ as $[\gamma, \gamma]$. When $\gamma=0$, which means the membership degree is zero, then we call it the empty set; While if $\gamma=1$, then we call it the full set. Note that we should not take the empty set as the set that there is no any value in it, and we also should not take the full set as the set of all possible values. This is the difference between the HFS and the traditional sets. The interpretations of these four HFEs in decision making are obvious. Consider an organization with several DMs from different areas to evaluate an alternative using HFEs. The empty set depicts that all DMs
oppose the alternative. The full set means that all DMs agree with it. The complete ignorance represents that all DMs have no idea for the alternative, and the nonsense set implies nonsense.

To compare the HFEs, Xia and Xu (2011a) defined the following comparison method:

Definition 1.2 (Xia and Xu 2011a). For a HFE $h, s(h)=\frac{1}{l_{h}} \sum_{\gamma \in h} \gamma$ is called the score of $h$, where $l_{h}$ is the number of the elements in $h$. For two HFEs $h_{1}$ and $h_{2}$, if $s\left(h_{1}\right)>s\left(h_{2}\right)$, then $h_{1}$ is superior to $h_{2}$, denoted by $h_{1}>h_{2}$; If $s\left(h_{1}\right)=s\left(h_{2}\right)$, then $h_{1}$ is indifferent to $h_{2}$, denoted by $h_{1} \sim h_{2}$.

Example 1.2. Let $h_{1}=\{0.2,0.4,0.5\}, h_{2}=\{0.3,0.4\}$ and $h_{3}=\{0.3,0.2,0.5,0.6\}$ be three HFEs, then by Definition 1.3, we have

$$
\begin{gathered}
s\left(h_{1}\right)=\frac{0.2+0.4+0.5}{3}=0.367 \\
s\left(h_{2}\right)=\frac{0.3+0.4}{2}=0.350 \\
s\left(h_{3}\right)=\frac{0.3+0.2+0.5+0.6}{4}=0.400
\end{gathered}
$$

Then $s\left(h_{3}\right)>s\left(h_{1}\right)>s\left(h_{2}\right)$, which indicates that $h_{3}>h_{1}>h_{2}$.
It is noted that $s(h)$ is directly related to the average value of all elements in $h$ expressing the average opinion of DMs. The higher the average value, the bigger the score $s(h)$, and thus, the better the HFE $h$. However, in some special cases, this comparison method cannot be used to distinguish two HFEs.

Example 1.3 (Liao et al. 2013). Let $h_{1}=\{0.1,0.1,0.7\}$ and $h_{2}=\{0.2,0.4\}$ be two HFEs, then by Definition 1.2, we have

$$
s\left(h_{1}\right)=\frac{0.1+0.1+0.7}{3}=0.3, s\left(h_{2}\right)=\frac{0.2+0.4}{2}=0.3
$$

Since $s\left(h_{1}\right)=s\left(h_{2}\right)$, we cannot tell the difference between $h_{1}$ and $h_{2}$ by using Definition 1.2. Actually, such a case is usually common in practice. It is
clear that Definition 1.2 does not take into account the situation that two HFEs $h_{1}$ and $h_{2}$ have the same score, but their deviation degrees may be different. The deviation degree of all elements with respect to the average value in a HFE reflects how these elements agree with each other, that is, they have a higher consistency. To better represent this issue, Chen et al. (2013a) defined the concept of deviation degree:

Definition 1.3 (Chen et al. 2013a). For a HFE $h$, we define the deviation degree $\bar{\sigma}(h)$ of $h$ as follows:

$$
\begin{equation*}
\bar{\sigma}(h)=\left[\frac{1}{l_{h}} \sum_{\gamma \in h}(\gamma-s(h))^{2}\right]^{\frac{1}{2}} \tag{1.2}
\end{equation*}
$$

As it can be seen that $s(h)$ is just as the mean value in statistics, and $\bar{\sigma}(h)$ is just as the standard variance, which reflects the deviation degree between all values in the HFE $h$ and their mean value. Inspired by this idea, based on the score $s(h)$ and the deviation degree $\bar{\sigma}(h)$, Chen et al. (2013a) gave a method for the comparison and the ranking of two HFEs below:

Definition 1.4 (Chen et al. 2013a). Let $h_{1}$ and $h_{2}$ be two HFEs, $s\left(h_{1}\right)$ and $s\left(h_{2}\right)$ the scores of $h_{1}$ and $h_{2}$, respectively, and $\bar{\sigma}\left(h_{1}\right)$ and $\bar{\sigma}\left(h_{2}\right)$ the deviation degrees of $h_{1}$ and $h_{2}$, respectively, then

- If $s\left(h_{1}\right)<s\left(h_{2}\right)$, then $h_{1}<h_{2}$.
- If $s\left(h_{1}\right)=s\left(h_{2}\right)$, then
(i) If $\bar{\sigma}\left(h_{1}\right)=\bar{\sigma}\left(h_{2}\right)$, then $h_{1}=h_{2}$.
(ii) If $\bar{\sigma}\left(h_{1}\right)<\bar{\sigma}\left(h_{2}\right)$, then $h_{1}>h_{2}$.
(iii) If $\bar{\sigma}\left(h_{1}\right)>\bar{\sigma}\left(h_{2}\right)$, then $h_{1}<h_{2}$.

Example 1.4 (Chen et al. 2013a). Let $h_{1}=\{0.2,0.3,0.5,0.8\}$, $h_{2}=\{0.4,0.6,0.8\}$ and $h_{3}=\{0.3,0.45,0.6\}$ be three HFEs, respectively. Since $\quad s\left(h_{1}\right)=0.45, \quad s\left(h_{2}\right)=0.6 \quad$ and $\quad s\left(h_{3}\right)=0.45$, then $s\left(h_{1}\right)=s\left(h_{3}\right)<s\left(h_{2}\right)$, and we get $h_{1}<h_{2}$ and $h_{3}<h_{2}$. Furthermore, $\bar{\sigma}\left(h_{1}\right)=0.229$ and $\bar{\sigma}\left(h_{3}\right)=0.123$, thus, $\bar{\sigma}\left(h_{1}\right)>\bar{\sigma}\left(h_{3}\right)$, and we have $h_{1}<h_{3}$. Therefore, $h_{1}<h_{3}<h_{2}$.

Additionally, Liao et al. (2013) developed another deviation degree as follows:
Definition 1.5 (Liao et al. 2013). Let $h$ be a HFE, then

$$
\begin{equation*}
\bar{\sigma}^{\prime}(h)=\frac{1}{l_{h}} \sqrt{\sum_{\gamma_{i}, \gamma_{j} \in h}\left(\gamma_{i}-\gamma_{j}\right)^{2}} \tag{1.3}
\end{equation*}
$$

is called the deviation degree of $h$, which reflects the standard deviation among all pairs of elements in the HFE $h$.
For two HFEs $h_{1}$ and $h_{2}$, if $\bar{\sigma}^{\prime}\left(h_{1}\right)>\bar{\sigma}^{\prime}\left(h_{2}\right)$, then $h_{1}<h_{2}$; If $\bar{\sigma}^{\prime}\left(h_{1}\right)=\bar{\sigma}^{\prime}\left(h_{2}\right)$, then $h_{1}=h_{2}$.

By Definition 1.5, we can obtain in Example 1.3 that

$$
\bar{\sigma}^{\prime}\left(h_{1}\right)=\frac{\sqrt{0+0.6^{2}+0.6^{2}}}{3}=0.283, \bar{\sigma}^{\prime}\left(h_{2}\right)=\frac{\sqrt{0.2^{2}}}{2}=0.100
$$

Then $\bar{\sigma}^{\prime}\left(h_{1}\right)>\bar{\sigma}^{\prime}\left(h_{2}\right)$, i.e., the variance degree of $h_{1}$ is higher than that of $h_{2}$, and thus, $h_{1}<h_{2}$.

### 1.1.2 Basic Operations and Relations

Given three HFEs represented by $h, h_{1}$ and $h_{2}$, Torra and Narukawa (2009), and Torra (2010) defined some operations on them, which can be described as:
(1) $h^{c}=\bigcup_{\gamma \in h}\{1-\gamma\}$.
(2) $h_{1} \bigcup h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}$.
(3) $h_{1} \bigcap h_{2}=\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}{\bigcup} \min \left\{\gamma_{1}, \gamma_{2}\right\}$.

Given an IFN (Xu and Yager 2006; Xu 2007a) $(\mu, v)$, its corresponding HFE is straightforward: $h=[\mu, 1-v]$, if $\mu \neq 1-v$. But, the construction of the IFN from the HFE is not so easy when the HFE contains more than one number for each $x \in X$. As for this issue, Torra and Narukawa (2009), and Torra (2010) showed that the envelop of a HFE is an IFN, expressed in the following definition:

Definition 1.6 (Torra and Narukawa 2009; Torra 2010). Given a HFE $h$, the IFN $\alpha_{\text {env }}(h)$ is defined as the envelope of $h$, where $\alpha_{\text {env }}(h)$ can be represented as $\left(h^{-}, 1-h^{+}\right)$, with $h^{-}=\min \{\gamma \mid \gamma \in h\}$ and $h^{+}=\max \{\gamma \mid \gamma \in h\}$ being its lower and upper bounds, respectively.

Then, Torra (2010) gave the further study of the relationship between HFEs and IFNs:

Theorem 1.1 (Torra 2010).
(1) $\alpha_{\text {env }}\left(h^{c}\right)=\left(\alpha_{\text {env }}(h)\right)^{c}$.
(2) $\alpha_{\text {env }}\left(h_{1} \cup h_{2}\right)=\alpha_{\text {env }}\left(h_{1}\right) \bigcup \alpha_{\text {env }}\left(h_{2}\right)$.
(3) $\alpha_{\text {env }}\left(h_{1} \cap h_{2}\right)=\alpha_{\text {env }}\left(h_{1}\right) \cap \alpha_{\text {env }}\left(h_{2}\right)$.

A similar theorem holds when $h_{1}$ and $h_{2}$ define intervals:
Theorem 1.2 (Torra 2010). Let $h_{1}$ and $h_{2}$ be two HFEs with $h(x)$ being a nonempty convex set for all $x$ in $X$, i.e., $h_{1}$ and $h_{2}$ are IFNs, then
(1) $h_{1}^{c}$ is equivalent to the IFN complement.
(2) $h_{1} \bigcap h_{2}$ is equivalent to the IFN intersection.
(3) $h_{1} \cup h_{2}$ is equivalent to the IFN union.

The above theorem reveals that the operations defined for HFEs are consistent with the ones for IFNs.

Based on the relationship between HFEs and IFNs, Xia and Xu (2011a) defined some operations for HFEs:

Definition 1.7 (Xia and Xu 2011a). Let $h, h_{1}$ and $h_{2}$ be three HFEs, and $\lambda>0$, then
(1) $h^{\lambda}=\bigcup_{\gamma \in h}\left\{\gamma^{\lambda}\right\}$.
(2) $\lambda h=\bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda}\right\}$.
(3) $h_{1} \oplus h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2}\right\}$.
(4) $h_{1} \otimes h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1} \gamma_{2}\right\}$.

In fact, all the above operations on HFEs can be suitable for IFNs. However, neither Torra (2010) nor Xia and Xu (2011a) paid any attention on the subtraction and division operations over HFEs (or IFNs), which are significantly important in forming the integral theoretical framework of hesitant fuzzy information (Liao and Xu 2013b). Therefore, Liao and Xu (2013b) introduced the subtraction and division operations for HFEs:

Definition 1.8 (Liao and Xu 2013b). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $h_{1} \ominus h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\{\bar{\gamma}\}$, where

$$
\bar{\gamma}= \begin{cases}\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}, & \text { if } \gamma_{1} \geq \gamma_{2} \text { and } \gamma_{2} \neq 1  \tag{1.4}\\ 0, & \text { otherwise }\end{cases}
$$

(2) $h_{1} \oslash h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\{\bar{\gamma}\}$, where

$$
\bar{\gamma}= \begin{cases}\frac{\gamma_{1}}{\gamma_{2}}, & \text { if } \gamma_{1} \leq \gamma_{2} \text { and } \gamma_{2} \neq 0  \tag{1.5}\\ 1, & \text { otherwise }\end{cases}
$$

To make it more adequate, Liao and Xu (2013b) further let:
(3) $h \ominus U^{*}=O^{*}$.
(4) $h \oslash U^{*}=O^{*}$.

From which it is obvious that for any HFE $h$, the following equations hold:
(5) $h \ominus h=O^{*}$.
(6) $h \ominus O^{*}=h$.
(7) $h \ominus I^{*}=O^{*}$.
(8) $h \oslash h=I^{*}$.
(9) $h \oslash I^{*}=h$.
(10) $h \oslash O^{*}=I^{*}$.

In addition, it follows from the above equations that some special cases hold:
(1) $I^{*} \ominus I^{*}=O^{*} ; U^{*} \ominus I^{*}=O^{*} ; O^{*} \ominus I^{*}=O^{*}$.
(2) $I^{*} \ominus U^{*}=O^{*} ; U^{*} \ominus U^{*}=O^{*} ; O^{*} \ominus U^{*}=O^{*}$.
(3) $I^{*} \Theta O^{*}=I^{*} ; U^{*} \Theta O^{*}=U^{*} ; O^{*} \Theta O^{*}=O^{*}$.
(4) $I^{*} \oslash I^{*}=I^{*} ; U^{*} \oslash I^{*}=U^{*} ; O^{*} \oslash I^{*}=O^{*}$.
(5) $I^{*} \oslash U^{*}=O^{*} ; U^{*} \oslash U^{*}=O^{*} ; O^{*} \oslash U^{*}=O^{*}$.
(6) $I^{*} \oslash O^{*}=I^{*} ; U^{*} \oslash O^{*}=I^{*} ; O^{*} \oslash O^{*}=I^{*}$.

For the brevity of presentation, in the process of theoretical derivation thereafter, we shall not consider the particular case where $\bar{\gamma}=0$ in the subtraction operation and $\bar{\gamma}=1$ in the division operation.

Similar to Theorem 1.1, the relationship between IFNs and HFEs can be further discussed:

Theorem 1.3 (Xia and Xu 2011a). Let $h, h_{1}$ and $h_{2}$ be three HFEs, and $\lambda>0$, then
(1) $\alpha_{\text {env }}\left(h^{\lambda}\right)=\left(\alpha_{\text {env }}(h)\right)^{\lambda}$.
(2) $\alpha_{\text {env }}(\lambda h)=\lambda\left(\alpha_{\text {env }}(h)\right)$.
(3) $\alpha_{\text {env }}\left(h_{1} \oplus h_{2}\right)=\alpha_{\text {env }}\left(h_{1}\right) \oplus \alpha_{\text {env }}\left(h_{2}\right)$.
(4) $\alpha_{\text {env }}\left(h_{1} \otimes h_{2}\right)=\alpha_{\text {env }}\left(h_{1}\right) \otimes \alpha_{\text {env }}\left(h_{2}\right)$.

Proof. For any three HFEs $h, h_{1}$ and $h_{2}$, we have
(1) $\alpha_{e n v}\left(h^{\lambda}\right)=\alpha_{e n v}\left(\left\{\gamma^{\lambda} \mid \gamma \in h\right\}\right)=\left(\left(h^{-}\right)^{\lambda}, 1-\left(h^{+}\right)^{\lambda}\right)$,
$\left(\alpha_{\text {env }}(h)\right)^{\lambda}=\left(h^{-}, 1-h^{+}\right)^{\lambda}=\left(\left(h^{-}\right)^{\lambda}, 1-\left(1-\left(1-h^{+}\right)\right)^{\lambda}\right)=\left(\left(h^{-}\right)^{\lambda}, 1-\left(h^{+}\right)^{\lambda}\right)$.
(2) $\alpha_{\text {env }}(\lambda h)=\alpha_{\text {env }}\left(\left\{1-(1-\gamma)^{\lambda} \mid \gamma \in h\right\}\right)$

$$
\begin{gathered}
=\left(1-\left(1-h^{-}\right)^{\lambda}, 1-\left(1-\left(1-h^{+}\right)^{\lambda}\right)\right)=\left(1-\left(1-h^{-}\right)^{\lambda},\left(1-h^{+}\right)^{\lambda}\right) \\
\lambda\left(\alpha_{e n v}(h)\right)=\lambda\left(h^{-}, 1-h^{+}\right)=\left(1-\left(1-h^{-}\right)^{\lambda},\left(1-h^{+}\right)^{\lambda}\right) .
\end{gathered}
$$

(3) $\alpha_{e n v}\left(h_{1} \oplus h_{2}\right)=\alpha_{e n v}\left(\left\{\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2} \mid \gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}\right\}\right)$
$=\left(h_{1}^{-}+h_{2}^{-}-h_{1}^{-} h_{2}^{-}, 1-\left(h_{1}^{+}+h_{2}^{+}-h_{1}^{+} h_{2}^{+}\right)\right)=\left(h_{1}^{-}+h_{2}^{-}-h_{1}^{-} h_{2}^{-},\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)\right)$,
$\alpha_{e n v}\left(h_{1}\right) \oplus \alpha_{e n v}\left(h_{2}\right)=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \oplus\left(h_{2}^{-}, 1-h_{2}^{+}\right)=\left(h_{1}^{-}+h_{2}^{-}-h_{1}^{-} h_{2}^{-},\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)\right)$.
(4) $\alpha_{e n v}\left(h_{1} \otimes h_{2}\right)=\alpha_{e n v}\left(\left\{\gamma_{1} \gamma_{2} \mid \gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}\right\}\right)=\left(h_{1}^{-} h_{2}^{-}, 1-h_{1}^{+} h_{2}^{+}\right)$,
$\alpha_{e n v}\left(h_{1}\right) \otimes \alpha_{e n v}\left(h_{2}\right)=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \otimes\left(h_{2}^{-}, 1-h_{2}^{+}\right)$
$=\left(h_{1}^{-} h_{2}^{-},\left(1-h_{1}^{+}\right)+\left(1-h_{2}^{+}\right)-\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)\right)=\left(h_{1}^{-} h_{2}^{-}, 1-h_{1}^{+} h_{2}^{+}\right)$.

Thus the proof is completed.
Some relationships can also be further established for those operations on HFEs:

Theorem 1.4 (Xia and Xu 2011a). For three HFEs $h, h_{1}$ and $h_{2}$, then
(1) $h_{1}^{c} \bigcup h_{2}^{c}=\left(h_{1} \cap h_{2}\right)^{c}$.
(2) $h_{1}^{c} \cap h_{2}^{c}=\left(h_{1} \cup h_{2}\right)^{c}$.
(3) $\left(h^{c}\right)^{\lambda}=(\lambda h)^{c}$.
(4) $\lambda\left(h^{c}\right)=\left(h^{\lambda}\right)^{c}$.
(5) $h_{1}^{c} \oplus h_{2}^{c}=\left(h_{1} \otimes h_{2}\right)^{c}$.
(6) $h_{1}^{c} \otimes h_{2}^{c}=\left(h_{1} \oplus h_{2}\right)^{c}$.

Proof. For three HFEs $h, h_{1}$ and $h_{2}$, we have
(1) $h_{1}^{c} \cup h_{2}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{1-\gamma_{1}, 1-\gamma_{2}\right\}=\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}{\bigcup}\left\{1-\min \left\{\gamma_{1}, \gamma_{2}\right\}\right\}=\left(h_{1} \cap h_{2}\right)^{c}$.
(2) $h_{1}^{c} \cap h_{2}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{1-\gamma_{1}, 1-\gamma_{2}\right\}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\max \left\{\gamma_{1}, \gamma_{2}\right\}\right\}=\left(h_{1} \cup h_{2}\right)^{c}$.
(3) $\left(h^{c}\right)^{\lambda}=\bigcup_{\gamma \in h}\left\{(1-\gamma)^{\lambda}\right\}=\left(\bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda}\right\}\right)^{c}=(\lambda h)^{c}$.
(4) $\lambda h^{c}=\bigcup_{\gamma \in h}\left\{1-(1-(1-\gamma))^{\lambda}\right\}=\bigcup_{\gamma \in h}\left\{1-\gamma^{\lambda}\right\}=\left(h^{\lambda}\right)^{c}$.
(5) $h_{1}{ }^{c} \oplus h_{2}{ }^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\left(1-\gamma_{1}\right)+\left(1-\gamma_{2}\right)-\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right\}$

$$
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\gamma_{1} \gamma_{2}\right\}=\left(h_{1} \otimes h_{2}\right)^{c} .
$$

(6) $h_{1}^{c} \otimes h_{2}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right\}$

$$
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2}\right)\right\}=\left(h_{1} \oplus h_{2}\right)^{c},
$$

which completes the proof of the theorem.
Theorem 1.5 (Zhu et al. 2012b). Let $h, h_{1}$ and $h_{2}$ be three HFEs, $\lambda>0$, then
(1) $\lambda\left(h_{1} \oplus h_{2}\right)=\lambda h_{1} \oplus \lambda h_{2}$.
(2) $\left(h_{1} \otimes h_{2}\right)^{\lambda}=h_{1}^{\lambda} \otimes h_{2}^{\lambda}$.

Proof. By the operational of HFEs, we have
(1) Since

$$
\begin{align*}
& \lambda\left(h_{1} \oplus h_{2}\right)=\lambda\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2}\right\}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(1-\left(\gamma_{1}+\gamma_{2}-\gamma_{1} \gamma_{2}\right)^{\lambda}\right)\right\} \\
& \left.=\bigcup_{\gamma_{\in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(1-\left(1-\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)^{\lambda}\right)\right\}=\bigcup_{\gamma_{i} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(\left(1-\gamma_{1}\right)^{\lambda}\left(1-\gamma_{2}\right)^{\lambda}\right)\right\}} .1 .6\right) \tag{1.6}
\end{align*}
$$

and

$$
\begin{align*}
\lambda h_{1} \oplus \lambda h_{2} & =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \xi_{2}}\left\{1-\left(1-\gamma_{1}\right)^{\lambda}+1-\left(1-\gamma_{2}\right)^{\lambda}-\left(1-\left(1-\gamma_{1}\right)^{\lambda}\right)\left(1-\left(1-\gamma_{2}\right)^{\lambda}\right)\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(1-\gamma_{1}\right)^{\lambda}\left(1-\gamma_{2}\right)^{\lambda}\right\} \tag{1.7}
\end{align*}
$$

then we have $\lambda\left(h_{1} \oplus h_{2}\right)=\lambda h_{1} \oplus \lambda h_{2}$.
(2) Since

$$
\begin{equation*}
\left(h_{1} \otimes h_{2}\right)^{\lambda}=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1} \gamma_{2}\right\}\right)^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\left(\gamma_{1} \gamma_{2}\right)^{\lambda}\right\} \tag{1.8}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{1}^{\lambda} \otimes h_{2}^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}}\left\{\gamma_{1}^{\lambda}\right\} \otimes \bigcup_{\gamma_{2} \in h_{2}}\left\{\gamma_{2}^{\lambda}\right\}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\left(\gamma_{1} \gamma_{2}\right)^{\lambda}\right\} \tag{1.9}
\end{equation*}
$$

then we have $\left(h_{1} \otimes h_{2}\right)^{\lambda}=h_{1}^{\lambda} \otimes h_{2}^{\lambda}$, which completes the proof of the theorem.

Theorem 1.6 (Liao and Xu 2013b). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $\left(h_{1} \oplus h_{2}\right) \oplus h_{2}=h_{1}$, if $\gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1$.
(2) $\left(h_{1} \oslash h_{2}\right) \otimes h_{2}=h_{1}$, if $\gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0$.

Proof. For two HFEs $h_{1}$ and $h_{2}$, we have
(1) $\left(h_{1} \oplus h_{2}\right) \oplus h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}\right\} \oplus h_{2}$
$=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}+\gamma_{2}-\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}} \gamma_{2}\right\}$
$=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\gamma_{1}\left(1-\gamma_{2}\right)}{1-\gamma_{2}}\right\}=\bigcup_{\gamma_{1} \in h_{1}}\left\{\gamma_{1}\right\}=h_{1}$.
(2) $\left(h_{1} \oslash h_{2}\right) \otimes h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2}}\right\} \otimes h_{2}$ $=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2}} \cdot \gamma_{2}\right\}=\bigcup_{\gamma_{1} \in h_{1}}\left\{\gamma_{1}\right\}=h_{1}$.

This completes the proof of the theorem.

Theorem 1.7 (Liao and Xu 2013b). Let $h_{1}$ and $h_{2}$ be two HFEs, $\lambda>0$, then
(1) $\lambda\left(h_{1} \ominus h_{2}\right)=\lambda h_{1} \ominus \lambda h_{2}$, if $\gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1$.
(2) $\left(h_{1} \oslash h_{2}\right)^{\lambda}=h_{1}^{\lambda} \oslash h_{2}^{\lambda}$, if $\gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0$.

Proof. For two HFEs $h_{1}$ and $h_{2}$, we have
(1) $\lambda\left(h_{1} \ominus h_{2}\right)=\lambda\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}\right\}\right)$

$$
\begin{gathered}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{1-\left(1-\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}\right)^{\lambda}\right\} \\
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\left(1-\gamma_{2}\right)^{\lambda}-\left(1-\gamma_{1}\right)^{\lambda}}{\left(1-\gamma_{2}\right)^{\lambda}}\right\} . \\
\lambda h_{1} \Theta \lambda h_{2}=\bigcup_{\gamma_{1} \in h_{1}}\left\{1-\left(1-\gamma_{1}\right)^{\lambda}\right\} \Theta \bigcup_{\gamma_{2} \in h_{2}}\left\{1-\left(1-\gamma_{2}\right)^{\lambda}\right\} .
\end{gathered}
$$

Since $y=x^{\lambda}(\lambda>0)$ is a monotonically increasing function when $x>0$, and also since $\gamma_{1} \geq \gamma_{2}, \quad \gamma_{2} \neq 1$, it follows that $1-\left(1-\gamma_{1}\right)^{\lambda} \geq 1-\left(1-\gamma_{2}\right)^{\lambda}$, $1-\left(1-\gamma_{2}\right)^{\lambda} \neq 1$. Thus,

$$
\begin{gathered}
\lambda h_{1} \ominus \lambda h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\left(1-\left(1-\gamma_{1}\right)^{\lambda}\right)-\left(1-\left(1-\gamma_{2}\right)^{\lambda}\right)}{1-\left(1-\left(1-\gamma_{2}\right)^{\lambda}\right)}\right\} \\
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\left(1-\gamma_{2}\right)^{\lambda}-\left(1-\gamma_{1}\right)^{\lambda}}{\left.\left(1-\gamma_{2}\right)^{\lambda}\right\}=\lambda\left(h_{1} \ominus h_{2}\right)}\right. \\
\text { (2) }\left(h_{1} \oslash h_{2}\right)^{\lambda}=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2}}\right\}\right)^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\left(\frac{\gamma_{1}}{\gamma_{2}}\right)^{\lambda}\right\} . \\
h_{1}^{\lambda} \oslash{h_{2}}^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}}\left\{\gamma_{1}^{\lambda}\right\} \oslash \bigcup_{\gamma_{2} \in h_{2}}^{\bigcup}\left\{\gamma_{2}^{\lambda}\right\}
\end{gathered}
$$

Since $\gamma_{1} \leq \gamma_{2}$, and $\gamma_{2} \neq 0$, it yields $\gamma_{1}^{\lambda} \leq \gamma_{2}{ }^{\lambda}$, and $\gamma_{2}{ }^{\lambda} \neq 0$, then

$$
h_{1}^{\lambda} \oslash h_{2}^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\gamma_{1}^{\lambda}}{\gamma_{2}^{\lambda}}\right\}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\left(\frac{\gamma_{1}}{\gamma_{2}}\right)^{\lambda}\right\}=\left(h_{1} \oslash h_{2}\right)^{\lambda}
$$

This completes the proof of the theorem.

Theorem 1.8 (Liao and Xu 2013b). Let $h=\bigcup_{\gamma \in h}\{\gamma\}$ be a HFE, and $\lambda_{1} \geq \lambda_{2}>0$, then
(1) $\lambda_{1} h \ominus \lambda_{2} h=\left(\lambda_{1}-\lambda_{2}\right) h$, if $\gamma \neq 1$.
(2) $h^{\lambda_{1}} \oslash h^{\lambda_{2}}=h^{\left(\lambda_{1}-\lambda_{2}\right)}$, if $\gamma \neq 0$.

Proof. For a HFE $h$ and $\lambda_{1}, \lambda_{2}>0$, we have
(1) $\lambda_{1} h \ominus \lambda_{2} h=\bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda_{1}}\right\} \ominus \bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda_{2}}\right\}$.

Since $y=a^{x}(0<a<1)$ is a monotonically decreasing function when $x>0$, and also since $\lambda_{1} \geq \lambda_{2}, \quad \gamma \neq 1, \quad$ it follows that $1-(1-\gamma)^{\lambda_{1}} \geq 1-(1-\gamma)^{\lambda_{2}}, 1-(1-\gamma)^{\lambda_{2}} \neq 1$. Thus,
$\lambda_{1} h \oplus \lambda_{2} h=\bigcup_{\gamma \in h}\left\{\frac{\left(1-(1-\gamma)^{\lambda_{1}}\right)-\left(1-(1-\gamma)^{\lambda_{2}}\right)}{1-\left(1-(1-\gamma)^{\lambda_{2}}\right)}\right\}=\bigcup_{\gamma \in h}\left\{\frac{(1-\gamma)^{\lambda_{2}}-(1-\gamma)^{\lambda_{1}}}{(1-\gamma)^{\lambda_{2}}}\right\}$
$=\bigcup_{\gamma \in h}\left\{1-\frac{(1-\gamma)^{\lambda_{1}}}{(1-\gamma)^{\lambda_{2}}}\right\}=\bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda_{1}-\lambda_{2}}\right\}=\left(\lambda_{1}-\lambda_{2}\right) h$
(2) $h^{\lambda_{1}} \oslash h^{\lambda_{2}}=\bigcup_{\gamma \in h}\left\{\gamma^{\lambda_{1}}\right\} \oslash \bigcup_{\gamma \in h}\left\{\gamma^{\lambda_{2}}\right\}$.

From $\lambda_{1} \geq \lambda_{2}>0$, and $\gamma \neq 0$, we can obtain $\gamma^{\lambda_{1}} \leq \gamma^{\lambda_{2}}$, and $\gamma^{\lambda_{2}} \neq 0$. Hence,

$$
\begin{equation*}
h^{\lambda_{1}} \oslash h^{\lambda_{2}}=\bigcup_{\gamma \in h}\left\{\frac{\gamma^{\lambda_{1}}}{\gamma^{\lambda_{2}}}\right\}=\bigcup_{\gamma \in h}\left\{\gamma^{\left(\lambda_{1}-\lambda_{1}\right)}\right\}=h^{\left(\lambda_{1}-\lambda_{2}\right)} \tag{1.11}
\end{equation*}
$$

which completes the proof.
Theorem 1.9 (Liao and Xu 2013b). For three HFEs $h_{1}, h_{2}$, and $h_{3}$, the following are valid:
(1) $h_{1} \ominus h_{2} \ominus h_{3}=h_{1} \ominus h_{3} \ominus h_{2}$, if $\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0$.
(2) $h_{1} \oslash h_{2} \oslash h_{3}=h_{1} \oslash h_{3} \oslash h_{2}$, if $\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0$.

Proof. (1) Since $\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0$, then

$$
\begin{equation*}
\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}-\gamma_{3}=\frac{\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3}}{1-\gamma_{2}} \geq 0 \tag{1.12}
\end{equation*}
$$

Thus,

$$
\begin{aligned}
h_{1} \oplus h_{2} \ominus h_{3} & =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}\right\} \ominus h_{3} \\
& =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{2} \geq 0}\left\{\frac{\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}-\gamma_{3}}{1-\gamma_{3}}\right\} \\
& =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3}}{\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)}\right\}
\end{aligned}
$$

Also since $\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0$, then

$$
\begin{equation*}
\frac{\gamma_{1}-\gamma_{3}}{1-\gamma_{3}}-\gamma_{2}=\frac{\gamma_{1}-\gamma_{3}-\gamma_{2}+\gamma_{2} \gamma_{3}}{1-\gamma_{3}} \geq 0 \tag{1.13}
\end{equation*}
$$

Thus,

$$
\begin{align*}
h_{1} \oplus h_{3} \ominus h_{2} & =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{3} \in h_{3}, \gamma_{1} \geq \gamma_{3}, \gamma_{3} \neq 1}\left\{\frac{\gamma_{1}-\gamma_{3}}{1-\gamma_{3}}\right\} \Theta h_{2} \\
& =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\frac{\gamma_{1}-\gamma_{3}}{1-\gamma_{3}}-\gamma_{2}}{1-\gamma_{2}}\right\} \\
& =\underbrace{}_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3}}{\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)}\right\} \tag{1.14}
\end{align*}
$$

Thus, $h_{1} \ominus h_{2} \ominus h_{3}=h_{1} \ominus h_{3} \ominus h_{2}$.
(2) Since $\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0$, and $0<\gamma_{3} \leq 1$, then $\gamma_{1} \leq \gamma_{2} \gamma_{3} \leq \gamma_{2}$, and $\frac{\gamma_{1}}{\gamma_{2}} \leq \gamma_{3}$. Thus,

$$
\begin{align*}
h_{1} \oslash h_{2} \oslash h_{3} & =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2}}\right\} \oslash h_{3} \\
& =\bigcup_{\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1} / \gamma_{2}}{\gamma_{3}}\right\}=\bigcup_{\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2} \gamma_{3}}\right\} \tag{1.15}
\end{align*}
$$

Meanwhile, from $\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{3} \neq 0$, and $0<\gamma_{2} \leq 1$, we can also obtain $\gamma_{1} \leq \gamma_{2} \gamma_{3} \leq \gamma_{3}$, and $\frac{\gamma_{1}}{\gamma_{3}} \leq \gamma_{2}$. Thus,

$$
\begin{align*}
h_{1} \oslash h_{3} \oslash h_{2} & =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{3} \in h_{3}, \gamma_{1} \leq \gamma_{3}, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{3}}\right\} \oslash h_{2} \\
& =\bigcup_{\gamma_{1} \leq \gamma_{2}, \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1} / \gamma_{3}}{\gamma_{2}}\right\}=\bigcup_{\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2} \gamma_{3}}\right\} \tag{1.16}
\end{align*}
$$

Thus, $h_{1} \oslash h_{2} \oslash h_{3}=h_{1} \oslash h_{3} \oslash h_{2}$. This completes the proof of the theorem.

Theorem 1.10 (Liao and Xu 2013b). For three HFEs $h_{1}, h_{2}$, and $h_{3}$, the following are valid:
(1) $h_{1} \ominus h_{2} \ominus h_{3}=h_{1} \ominus\left(h_{2} \oplus h_{3}\right)$, if $\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1$, and $\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0$.
(2) $h_{1} \oslash h_{2} \oslash h_{3}=h_{1} \oslash\left(h_{2} \otimes h_{3}\right)$, if $\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0$, and $\gamma_{3} \neq 0$.

Proof. (1) According to (1) in Theorem 1.6, it follows that

$$
\begin{align*}
h_{1} \ominus h_{2} \ominus h_{3} & =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3}}{\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)}\right\}  \tag{1.17}\\
h_{1} \ominus\left(h_{2} \oplus h_{3}\right) & =h_{1} \ominus\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\gamma_{2}+\gamma_{3}-\gamma_{2} \gamma_{3}\right\}\right) \\
& =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\gamma_{1}-\left(\gamma_{2}+\gamma_{3}-\gamma_{2} \gamma_{3}\right)}{1-\left(\gamma_{2}+\gamma_{3}-\gamma_{2} \gamma_{3}\right)}\right\}
\end{align*}
$$

$$
\begin{align*}
& =\bigcup_{\gamma_{1} \geq \gamma_{2}, \gamma_{1} \geq \gamma_{3}, \gamma_{2} \neq 1, \gamma_{3} \neq 1, \gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3} \geq 0}\left\{\frac{\gamma_{1}-\gamma_{2}-\gamma_{3}+\gamma_{2} \gamma_{3}}{\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)}\right\} \\
& =h_{1} \ominus h_{2} \ominus h_{3} \tag{1.18}
\end{align*}
$$

(2) According to (2) in Theorem 1.6, we have

$$
\begin{array}{r}
h_{1} \oslash h_{2} \oslash h_{3}=\bigcup_{\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2} \gamma_{3}}\right\} \\
h_{1} \oslash\left(h_{2} \otimes h_{3}\right)=h_{1} \oslash\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\gamma_{2} \gamma_{3}\right\}\right) \\
=\bigcup_{\gamma_{1} \leq \gamma_{2} \gamma_{3}, \gamma_{2} \neq 0, \gamma_{3} \neq 0}\left\{\frac{\gamma_{1}}{\gamma_{2} \gamma_{3}}\right\}=h_{1} \oslash h_{2} \oslash h_{3} \tag{1.20}
\end{array}
$$

which completes the proof of the theorem.
It should be noted that in the above theorems, the equations hold only under the given preconditions. Moreover, the relationships between IFNs and HFEs can be further verified in terms of these two operations:

Theorem 1.11 (Liao and Xu 2013b). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $\alpha_{e n v}\left(h_{1} \ominus h_{2}\right)=\alpha_{e n v}\left(h_{1}\right) \ominus \alpha_{e n v}\left(h_{2}\right)$.
(2) $\alpha_{e n v}\left(h_{1} \oslash h_{2}\right)=\alpha_{e n v}\left(h_{1}\right) \oslash \alpha_{e n v}\left(h_{2}\right)$.

Proof. (1) $\alpha_{e n v}\left(h_{1} \ominus h_{2}\right)=\alpha_{e n v}\left(\left\{\left.\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}} \right\rvert\, \gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1\right\}\right)$

$$
=\left(\frac{h_{1}^{-}-h_{2}^{-}}{1-h_{2}^{-}}, 1-\frac{h_{1}^{+}-h_{2}^{+}}{1-h_{2}^{+}}\right)=\left(\frac{h_{1}^{-}-h_{2}^{-}}{1-h_{2}^{-}}, \frac{1-h_{1}^{+}}{1-h_{2}^{+}}\right) .
$$

$$
\begin{equation*}
\alpha_{e n v}\left(h_{1}\right) \ominus \alpha_{e n v}\left(h_{2}\right)=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \ominus\left(h_{2}^{-}, 1-h_{2}^{+}\right)=\left(\frac{h_{1}^{-}-h_{2}^{-}}{1-h_{2}^{-}}, \frac{1-h_{1}^{+}}{1-h_{2}^{+}}\right) \tag{1.21}
\end{equation*}
$$

(2) $\alpha_{e n v}\left(h_{1} \oslash h_{2}\right)=\alpha_{e n v}\left(\left\{\left.\frac{\gamma_{1}}{\gamma_{2}} \right\rvert\, \gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0\right\}\right)=\left(\frac{h_{1}^{-}}{h_{1}^{-}}, 1-\frac{h_{2}^{+}}{h_{2}^{+}}\right)$.

$$
\begin{align*}
\alpha_{e n v}\left(h_{1}\right) \oslash \alpha_{e n v}\left(h_{2}\right) & =\left(h_{1}^{-}, 1-h_{1}^{+}\right) \oslash\left(h_{2}^{-}, 1-h_{2}^{+}\right) \\
& =\left(\frac{h_{1}^{-}}{h_{1}^{-}}, \frac{\left(1-h_{1}^{+}\right)-\left(1-h_{2}^{+}\right)}{1-\left(1-h_{2}^{+}\right)}\right)=\left(\frac{h_{1}^{-}}{h_{1}^{-}}, 1-\frac{h_{1}^{+}}{h_{2}^{+}}\right) \tag{1.22}
\end{align*}
$$

Thus, the proof is completed.
Theorem 1.11 further reveals that the subtraction and division operations defined for HFEs are consistent with the ones for IFNs. The following theorem reveals the relationship between these two operations:

Theorem 1.12 (Liao and Xu 2013b). For two HFEs $h_{1}$ and $h_{2}$, the following are valid:
(1) $h_{1}^{c} \Theta h_{2}^{c}=\left(h_{1} \oslash h_{2}\right)^{c}$.
(2) $h_{1}^{c} \oslash h_{2}^{c}=\left(h_{1} \Theta h_{2}\right)^{c}$.

Proof. (1) $h_{1}^{c} \ominus h_{2}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\left(1-\gamma_{1}\right)-\left(1-\gamma_{2}\right)}{1-\left(1-\gamma_{2}\right)}\right\}$
$=\bigcup_{\gamma_{\in} \in h_{1}, \gamma_{2} \leq h_{2}, \gamma_{1} \leqslant \gamma_{2}, \gamma_{2} \neq 0}\left\{\frac{\left(1-\gamma_{1}\right)-\left(1-\gamma_{2}\right)}{1-\left(1-\gamma_{2}\right)}\right\}$
$=\bigcup_{\gamma_{i \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \leq \gamma_{2}, \gamma_{2} \neq 0}}\left\{1-\frac{\gamma_{1}}{\gamma_{2}}\right\}=\left(h_{1} \oslash h_{2}\right)^{c}$.
(2) $h_{1}^{c} \oslash h_{2}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{\frac{1-\gamma_{1}}{1-\gamma_{2}}\right\}$
$=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{1} \geq \gamma_{2}, \gamma_{2} \neq 1}\left\{1-\frac{\gamma_{1}-\gamma_{2}}{1-\gamma_{2}}\right\}=\left(h_{1} \Theta h_{2}\right)^{c}$,
which completes the proof of the theorem.

Example 1.5 (Liao and Xu 2013b). Consider two HFEs $h_{1}=\{0.3,0.2\}$ and $h_{2}=\{0.1,0.2\}$. According to (1) in Definition 1.8, we have

$$
h_{1} \ominus h_{2}=\left\{\frac{0.3-0.1}{1-0.1}, \frac{0.3-0.2}{1-0.2}, \frac{0.2-0.1}{1-0.1}, \frac{0.2-0.2}{1-0.2}\right\}=\left\{\frac{2}{9}, \frac{1}{8}, \frac{1}{9}, 0\right\}
$$

In addition, $h_{1}^{c}=\{0.7,0.8\}$, and $h_{2}^{c}=\{0.9,0.8\}$. Then by (2) in Definition 1.8, we obtain

$$
h_{1}^{c} \oslash h_{2}^{c}=\left\{\frac{0.7}{0.9}, \frac{0.8}{0.9}, \frac{0.7}{0.8}, \frac{0.8}{0.8}\right\}=\left\{\frac{7}{9}, \frac{8}{9}, \frac{7}{8}, 1\right\}
$$

Since

$$
\left(h_{1} \ominus h_{2}\right)^{c}=\left\{1-\frac{2}{9}, 1-\frac{1}{8}, 1-\frac{1}{9}, 1-0\right\}=\left\{\frac{7}{9}, \frac{8}{9}, \frac{7}{8}, 1\right\}
$$

then $\left(h_{1} \ominus h_{2}\right)^{c}=h_{1}{ }^{c} \oslash h_{2}{ }^{c}$, which verifies (2) of Theorem 1.12. (1) of Theorem 1.12 can be verified similarly.

Now we introduce the concepts of t -norms and t -conorms which are defined as follows:

Definition 1.9 (Klir and Yuan 1995; Nguyen and Walker 1997). A function $\dot{T}$ : $[0,1] \times[0,1] \rightarrow[0,1]$ is called a t-norm if it satisfies the following four conditions:
(1) $\dot{T}(1, x)=x$, for all $x$.
(2) $\dot{T}(x, y)=\dot{T}(y, x)$, for all $x$ and $y$.
(3) $\dot{T}(x, \dot{T}(y, z))=\dot{T}(\dot{T}(x, y), z)$, for all $x, y$ and $z$.
(4) If $x \leq x^{\prime}$ and $y \leq y^{\prime}$, then $\dot{T}(x, y) \leq \dot{T}\left(x^{\prime}, y^{\prime}\right)$.

Definition 1.10 (Klir and Yuan 1995; Nguyen and Walker 1997). A function $\dot{S}$ : $[0,1] \times[0,1] \rightarrow[0,1]$ is called a $t$-conorm if it satisfies the following four conditions:
(1) $\dot{S}(0, x)=x$, for all $x$.
(2) $\dot{S}(x, y)=\dot{S}(y, x)$, for all $x$ and $y$.
(3) $\dot{S}(x, \dot{S}(y, z))=\dot{S}(\dot{S}(x, y), z)$, for all $x, y$ and $z$.
(4) If $x \leq x^{\prime}$ and $y \leq y^{\prime}$, then $\dot{S}(x, x) \leq \dot{S}\left(x^{\prime}, x^{\prime}\right)$.

Definition 1.11 (Klir and Yuan 1995; Nguyen and Walker 1997). A t-norm function $\dot{T}(x, y)$ is called Archimedean t -norm if it is continuous and $\dot{T}(x, x)<x$ for all $x \in(0,1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in(0,1)$.

Definition 1.12 (Klir and Yuan 1995; Nguyen and Walker 1997). A t-conorm function $\dot{S}(x, y)$ is called Archimedean t-conorm if it is continuous and $\dot{S}(x, x)>x$ for all $x \in(0,1)$. An Archimedean t -conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in(0,1)$.

It is well known (Klement and Mesiar 2005) that a strict Archimedean t-norm is expressed via its additive generator $\dot{\tau}$ as $\dot{T}(x, y)=\dot{\tau}^{-1}(\dot{\tau}(x)+\dot{\tau}(y))$, and similarly, applied to the t-conorm $\dot{S}(x, y)=\dot{s}^{-1}(\dot{s}(x)+\dot{s}(y))$ with $\dot{s}(t)=\dot{\tau}(1-t)$. Moreover, the additive generator of a continuous Archimedean t-norm is a strictly decreasing function $\dot{\tau}:[0,1] \rightarrow[0, \infty]$ such that $\dot{\tau}(1)=0$.

Based on the relationship between HFEs and IFNs, Xia and Xu (2012a) defined some general operations for the HFEs based on Archimedean t-norm and t-conorm (Klir and Yuan 1995; Nguyen and Walker 1997):

Definition 1.13 (Xia and Xu 2012a). Let $h, h_{1}$ and $h_{2}$ be three HFEs, then
(1) $h^{\lambda}=\bigcup_{\gamma \in h}\left\{\dot{\tau}^{-1}(\lambda \dot{\tau}(\gamma))\right\}$.
(2) $\lambda h=\bigcup_{\gamma \in h}\left\{\dot{s}^{-1}(\lambda \dot{s}(\gamma))\right\}$.
(3) $h_{1} \otimes h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{2}\right)\right)\right\}$.
(4) $h_{1} \oplus h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right\}$.
where $\dot{s}(t)=\dot{\tau}(1-t)$, and $\dot{\tau}:[0,1] \rightarrow[0, \infty]$ is a strictly decreasing function such that $\dot{\tau}(1)=0$.

Based on the operations in Definition 1.13, we can prove that Theorems 1.3 and 1.4 also hold. Moreover, some other relationships can be established for these operations on HFEs:

Theorem 1.13 (Xia and Xu 2012a). For three HFEs $h_{1}, h_{2}$ and $h_{3}$, the following are valid:
(1) $h_{1} \oplus\left(h_{2} \oplus h_{3}\right)=\left(h_{1} \oplus h_{2}\right) \oplus h_{3}$.
(2) $h_{1} \otimes\left(h_{2} \otimes h_{3}\right)=\left(h_{1} \otimes h_{2}\right) \otimes h_{3}$.

Proof. (1) $h_{1} \oplus\left(h_{2} \oplus h_{3}\right)=h_{1} \oplus \underset{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}{\bigcup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}^{\bigcup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(s^{-1}\left(\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right)\right)\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}^{\bigcup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}^{\bigcup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\} \\
& =\left(h_{1} \oplus h_{2}\right) \oplus h_{3} .
\end{aligned}
$$

Similarly, (2) can be proven.
Theorem 1.14 (Xia and Xu 2012a). Let $h_{1}, h_{2}$ and $h_{3}$ be three HFEs, then
(1) $\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \oplus h_{3}\right)\right)=\alpha_{e n v}\left(h_{1}\right) \oplus\left(\alpha_{e n v}\left(h_{2}\right) \oplus \alpha_{e n v}\left(h_{3}\right)\right)$.
(2) $\alpha_{e n v}\left(\left(h_{1} \oplus h_{2}\right) \oplus h_{3}\right)=\left(\alpha_{e n v}\left(h_{1}\right) \oplus \alpha_{e n v}\left(h_{2}\right)\right) \oplus \alpha_{e n v}\left(h_{3}\right)$.
(3) $\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \oplus h_{3}\right)\right)=\alpha_{e n v}\left(\left(h_{1} \oplus h_{2}\right) \oplus h_{3}\right)$.
(4) $\alpha_{e n v}\left(h_{1}\right) \oplus\left(\alpha_{e n v}\left(h_{2}\right) \oplus \alpha_{e n v}\left(h_{3}\right)\right)=\left(\alpha_{e n v}\left(h_{1}\right) \oplus \alpha_{e n v}\left(h_{2}\right)\right) \oplus \alpha_{e n v}\left(h_{3}\right)$.
(5) $\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \otimes h_{3}\right)\right)=\alpha_{e n v}\left(h_{1}\right) \otimes\left(\alpha_{e n v}\left(h_{2}\right) \otimes \alpha_{e n v}\left(h_{3}\right)\right)$.
(6) $\alpha_{e n v}\left(\left(h_{1} \otimes h_{2}\right) \otimes h_{3}\right)=\left(\alpha_{e n v}\left(h_{1}\right) \otimes \alpha_{e n v}\left(h_{2}\right)\right) \otimes \alpha_{e n v}\left(h_{3}\right)$.
(7) $\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \otimes h_{3}\right)\right)=\alpha_{e n v}\left(\left(h_{1} \otimes h_{2}\right) \otimes h_{3}\right)$.
(8) $\alpha_{e n v}\left(h_{1}\right) \otimes\left(\alpha_{e n v}\left(h_{2}\right) \otimes \alpha_{e n v}\left(h_{3}\right)\right)=\left(\alpha_{e n v}\left(h_{1}\right) \otimes \alpha_{e n v}\left(h_{2}\right)\right) \otimes \alpha_{e n v}\left(h_{3}\right)$.

Proof. We only prove (1) and (5), the others can be obtained similarly:

$$
\begin{aligned}
& \text { (1) } \alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \oplus h_{3}\right)\right) \\
& =\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{1}^{-}\right)\right), 1-\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{1}^{+}\right)\right)\right)
\end{aligned}
$$

Since $\dot{s}(t)=\dot{\tau}(1-t)$, and $\dot{S}^{-1}(t)=1-\dot{\tau}^{-1}(t)$, then

$$
\begin{align*}
\alpha_{e n v} & \left(h_{1} \oplus\left(h_{2} \oplus h_{3}\right)\right) \\
& =\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{1}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{1}^{+}\right)\right)\right) \tag{1.23}
\end{align*}
$$

On the other hand, we have

$$
\begin{aligned}
\alpha_{e n v} & \left(h_{1}\right) \oplus\left(\alpha_{e n v}\left(h_{2}\right) \oplus \alpha_{e n v}\left(h_{3}\right)\right)=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \oplus\left(\left(h_{2}^{-}, 1-h_{2}^{+}\right) \oplus\left(h_{3}^{-}, 1-h_{3}^{+}\right)\right) \\
& =\left(h_{1}^{-}, 1-h_{1}^{+}\right) \oplus\left(\dot{s}^{-1}\left(\dot{s}\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{2}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right)\right) \\
& =\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{2}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right)\right)
\end{aligned}
$$

which derives that

$$
\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \oplus h_{3}\right)\right)=\alpha_{e n v}\left(h_{1}\right) \oplus\left(\alpha_{e n v}\left(h_{2}\right) \oplus \alpha_{e n v}\left(h_{3}\right)\right)
$$

(5) $\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \otimes h_{3}\right)\right)$

$$
=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), 1-\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{+}\right)+\dot{\tau}\left(h_{2}^{+}\right)+\dot{\tau}\left(h_{3}^{+}\right)\right)\right) .
$$

Since $\dot{\tau}(t)=\dot{s}(1-t)$, and $\dot{\tau}^{-1}(t)=1-\dot{s}^{-1}(t)$, then

$$
\begin{aligned}
& \alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \otimes h_{3}\right)\right) \\
& \quad=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{1}^{+}\right)+\dot{s}\left(1-h_{2}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& \alpha_{e n v}\left(h_{1}\right) \otimes\left(\alpha_{e n v}\left(h_{2}\right) \otimes \alpha_{e n v}\left(h_{3}\right)\right)=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \otimes\left(\left(h_{2}^{-}, 1-h_{2}^{+}\right) \otimes\left(h_{3}^{-}, 1-h_{3}^{+}\right)\right) \\
&=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \otimes\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{2}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right)
\end{aligned}
$$

$$
=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{1}^{+}\right)+\dot{s}\left(1-h_{2}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right)
$$

which derives that

$$
\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \otimes h_{3}\right)\right)=\alpha_{e n v}\left(h_{1}\right) \otimes\left(\alpha_{e n v}\left(h_{2}\right) \otimes \alpha_{e n v}\left(h_{3}\right)\right)
$$

Theorem 1.15 (Xia and Xu 2012a). Let $h_{1}, h_{2}$ and $h_{3}$ be three HFEs, then
(1) $\left(h_{1} \cup h_{2}\right) \oplus h_{3}=\left(h_{1} \oplus h_{3}\right) \bigcup\left(h_{2} \oplus h_{3}\right)$.
(2) $\left(h_{1} \cap h_{2}\right) \oplus h_{3}=\left(h_{1} \oplus h_{3}\right) \bigcap\left(h_{2} \oplus h_{3}\right)$.
(3) $\left(h_{1} \cup h_{2}\right) \otimes h_{3}=\left(h_{1} \otimes h_{3}\right) \cup\left(h_{2} \otimes h_{3}\right)$.
(4) $\left(h_{1} \cap h_{2}\right) \otimes h_{3}=\left(h_{1} \otimes h_{3}\right) \cap\left(h_{2} \otimes h_{3}\right)$.
(5) $h_{1} \oplus\left(h_{2} \cup h_{3}\right)=\left(h_{1} \oplus h_{2}\right) \bigcup\left(h_{1} \oplus h_{3}\right)$.
(6) $h_{1} \oplus\left(h_{2} \bigcap h_{3}\right)=\left(h_{1} \oplus h_{2}\right) \bigcap\left(h_{1} \oplus h_{3}\right)$.
(7) $h_{1} \otimes\left(h_{2} \cup h_{3}\right)=\left(h_{1} \otimes h_{2}\right) \bigcup\left(h_{1} \otimes h_{3}\right)$.
(8) $h_{1} \otimes\left(h_{2} \cap h_{3}\right)=\left(h_{1} \otimes h_{2}\right) \bigcap\left(h_{1} \otimes h_{3}\right)$.

Proof. In the following, we prove (1), (3), (5) and (7), others can be proven similarly:
(1) $\left(h_{1} \cup h_{2}\right) \oplus h_{3}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\} \oplus h_{3}$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{s}^{-1}\left(\dot{s}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{s}\left(\gamma_{3}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{s}^{-1}\left(\max \left\{s\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right), \dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left(\max \left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(\bigcup_{\gamma_{\in} \in h_{1}, \gamma_{3} \in h_{3}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \cup\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}^{\cup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(h_{1} \oplus h_{3}\right) \cup\left(h_{2} \oplus h_{3}\right) .
\end{aligned}
$$

(3) $\left(h_{1} \cup h_{2}\right) \otimes h_{3}=\left(\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}{\bigcup} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \otimes h_{3}$

$$
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{\tau}^{-1}\left(\dot{\tau}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)
$$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{\tau}^{-1}\left(\max \left\{\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right), \dot{\tau}\left(\gamma_{2}\right)+\dot{\tau}\left(\gamma_{3}\right)\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left(\max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{2}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{3} \in h_{3}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \bigcup\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{2}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(h_{1} \otimes h_{3}\right) \bigcup\left(h_{2} \otimes h_{3}\right) .
\end{aligned}
$$

(5) $h_{1} \oplus\left(h_{2} \cup h_{3}\right)=h_{1} \oplus\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \max \left\{\gamma_{2}, \gamma_{3}\right\}\right)$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\max \left\{\gamma_{2}, \gamma_{3}\right\}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{s}^{-1}\left(\max \left\{\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right), \dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left(\max \left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right\}\right) \bigcup\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{3} \in h_{3}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(h_{1} \oplus h_{3}\right) \bigcup\left(h_{1} \oplus h_{2}\right) .
\end{aligned}
$$

(7) $h_{1} \otimes\left(h_{2} \cup h_{3}\right)=h_{1} \otimes\left(\bigcup_{\gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \max \left\{\gamma_{2}, \gamma_{3}\right\}\right)$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\max \left\{\gamma_{2}, \gamma_{3}\right\}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}} \dot{\tau}^{-1}\left(\max \left\{\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{2}\right), \dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left(\max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{2}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{2}\right)\right)\right\}\right) \bigcup\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{3} \in h_{3}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \\
& =\left(h_{1} \otimes h_{3}\right) \bigcup\left(h_{1} \otimes h_{2}\right) .
\end{aligned}
$$

Theorem 1.16 (Xia and Xu 2012a). Let $h_{1}, h_{2}$ and $h_{3}$ be three HFEs, then
(1) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus h_{3}\right) \cup \alpha_{e v n}\left(h_{2} \oplus h_{3}\right)$.
(2) $\alpha_{e n v}\left(\left(h_{1} \cap h_{2}\right) \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus h_{3}\right) \cap \alpha_{e v n}\left(h_{2} \oplus h_{3}\right)$.
(3) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes h_{3}\right) \cup \alpha_{e v n}\left(h_{2} \otimes h_{3}\right)$.
(4) $\alpha_{e n v}\left(\left(h_{1} \cap h_{2}\right) \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes h_{3}\right) \cap \alpha_{e n v}\left(h_{2} \otimes h_{3}\right)$.
(5) $\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \cup h_{3}\right)\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right) \cup \alpha_{e v n}\left(h_{1} \oplus h_{3}\right)$.
(6) $\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \cap h_{3}\right)\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right) \cap \alpha_{e n v}\left(h_{1} \oplus h_{3}\right)$.
(7) $\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \cup h_{3}\right)\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right) \cup \alpha_{e n v}\left(h_{1} \otimes h_{3}\right)$.
(8) $\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \cap h_{3}\right)\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right) \cap \alpha_{e n v}\left(h_{1} \otimes h_{3}\right)$.
(9) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \cup h_{3}\right)\right)$.
(10) $\alpha_{\text {env }}\left(\left(h_{1} \cap h_{2}\right) \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus\left(h_{2} \cap h_{3}\right)\right)$.
(11) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \cup h_{3}\right)\right)$.
(12) $\alpha_{e n v}\left(\left(h_{1} \cap h_{2}\right) \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes\left(h_{2} \cap h_{3}\right)\right)$.
(13) $\alpha_{e n v}\left(h_{1} \oplus h_{3}\right) \cup \alpha_{e n v}\left(h_{2} \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right) \cup \alpha_{e v n}\left(h_{1} \oplus h_{3}\right)$.
(14) $\alpha_{e n v}\left(h_{1} \oplus h_{3}\right) \cap \alpha_{e n v}\left(h_{2} \oplus h_{3}\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right) \cap \alpha_{e v n}\left(h_{1} \oplus h_{3}\right)$.
(15) $\alpha_{e n v}\left(h_{1} \otimes h_{3}\right) \cup \alpha_{e n v}\left(h_{2} \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right) \cup \alpha_{e v n}\left(h_{1} \otimes h_{3}\right)$.
(16) $\alpha_{e n v}\left(h_{1} \otimes h_{3}\right) \cap \alpha_{e n v}\left(h_{2} \otimes h_{3}\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right) \cap \alpha_{e v n}\left(h_{1} \otimes h_{3}\right)$.

Proof. We prove (1) and (3), others can be proven analogously:

$$
\text { (1) } \begin{aligned}
& \alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \oplus h_{3}\right)=\alpha_{e n v}\left(\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}{\cup}\left\{\dot{s}^{-1}\left(\dot{s}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
&= \alpha_{e n v}\left(\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}{\bigcup} \max \left\{\dot{s}^{-1}\left(s\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{3}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(\gamma_{2}\right)+\dot{s}\left(\gamma_{3}\right)\right)\right\}\right) \\
&=\left(\max \left\{\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right)\right\},\right. \\
&\left.1-\max \left\{\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{3}^{+}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(h_{2}^{+}\right)+\dot{s}\left(h_{3}^{+}\right)\right)\right\}\right) \\
&=\left(\max \left\{\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right)\right\},\right. \\
&\left.\min \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right), \tau^{-1}\left(\dot{\tau}\left(1-h_{2}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right)\right\}\right) \\
&=\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right)\right) \\
&\left.\left.\cup\left(\dot{s}^{-1}\left(\dot{s}\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{2}^{+}\right)+\dot{\tau}\left(1-h_{3}^{+}\right)\right)\right)+\dot{s}\left(h_{3}^{-}\right)\right), 1-\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{3}^{+}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& \cup\left(\dot{s}^{-1}\left(s\left(h_{2}^{-}\right)+\dot{s}\left(h_{3}^{-}\right)\right), 1-\dot{s}^{-1}\left(\dot{s}\left(h_{2}^{+}\right)+\dot{s}\left(h_{3}^{+}\right)\right)\right) \\
= & \alpha_{e n v}\left(h_{1} \oplus h_{3}\right) \cup \alpha_{e v n}\left(h_{2} \oplus h_{3}\right) .
\end{aligned}
$$

(3) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right) \otimes h_{3}\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right)$

$$
\begin{aligned}
= & \alpha_{e n v}\left(\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}{\cup} \max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{3}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{2}\right)+\dot{\tau}\left(\gamma_{3}\right)\right)\right\}\right) \\
= & \left(\max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right)\right\},\right. \\
& \left.1-\max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{+}\right)+\dot{\tau}\left(h_{3}^{+}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{+}\right)+\dot{\tau}\left(h_{3}^{+}\right)\right)\right\}\right) \\
= & \left(\max \left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right)\right\},\right. \\
& \left.\min \left\{\dot{s}^{-1}\left(\dot{s}\left(1-h_{1}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{2}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right\}\right) \\
= & \left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{1}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right) \\
& \cup\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{2}^{+}\right)+\dot{s}\left(1-h_{3}^{+}\right)\right)\right) \\
= & \left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), 1-\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{+}\right)+\dot{\tau}\left(h_{3}^{+}\right)\right)\right) \\
& \cup\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{-}\right)+\dot{\tau}\left(h_{3}^{-}\right)\right), 1-\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{2}^{+}\right)+\dot{\tau}\left(h_{3}^{+}\right)\right)\right) \\
= & \alpha_{e n v}\left(h_{1} \otimes h_{3}\right) \cup \alpha_{e v n}\left(h_{2} \otimes h_{3}\right) .
\end{aligned}
$$

Theorem 1.17 (Xia and Xu 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $\left(h_{1} \cup h_{2}\right) \oplus\left(h_{1} \cap h_{2}\right)=h_{1} \oplus h_{2}$.
(2) $\left(h_{1} \cup h_{2}\right) \otimes\left(h_{1} \cap h_{2}\right)=\left(h_{1} \otimes h_{2}\right)$.

Proof. (1) We know that for any two real numbers $a$ and $b$, it follows that:

$$
\max \{a, b\}+\min \{a, b\}=a+b, \max \{a, b\} \cdot \min \{a, b\}=a \cdot b
$$

then we have

$$
\left(h_{1} \cup h_{2}\right) \oplus\left(h_{1} \cap h_{2}\right)=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \oplus\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right)
$$

$$
\begin{aligned}
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \oplus\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \dot{s}^{-1}\left(\dot{s}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{s}\left(\min \left\{\gamma_{1}, \gamma_{2}\right\}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\max \left\{\left(\dot{s}\left(\gamma_{1}\right), \dot{s}\left(\gamma_{2}\right)\right\}+\min \left\{\dot{s}\left(\gamma_{1}\right), \dot{s}\left(\gamma_{2}\right)\right\}\right)\right\}\right. \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right\}=h_{1} \oplus h_{2}
\end{aligned}
$$

(2) $\left(h_{1} \cup h_{2}\right) \otimes\left(h_{1} \cap h_{2}\right)=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}^{\bigcup} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \otimes\left(\bigcup_{\gamma \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right)$

$$
\begin{aligned}
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \otimes\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} i^{-1}\left(\dot{\tau}\left(\max \left\{\gamma_{1}, \gamma_{2}\right\}\right)+\dot{\tau}\left(\min \left\{\gamma_{1}, \gamma_{2}\right\}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{\tau}^{-1}\left(\max \left\{\left(\dot{\tau}\left(\gamma_{1}\right), \dot{\tau}\left(\gamma_{2}\right)\right\}+\min \left\{\dot{\tau}\left(\gamma_{1}\right), \dot{\tau}\left(\gamma_{2}\right)\right\}\right)\right\}\right. \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\gamma_{1}\right)+\dot{\tau}\left(\gamma_{2}\right)\right)\right\}=h_{1} \otimes h_{2} .
\end{aligned}
$$

Theorem 1.18 (Xia and Xu 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $\alpha_{e n v}\left(h_{1} \cup h_{2}\right) \oplus \alpha_{e n v}\left(h_{1} \cap h_{2}\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right)$.
(2) $\alpha_{e n v}\left(h_{1} \cup h_{2}\right) \otimes \alpha_{e n v}\left(h_{1} \cap h_{2}\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right)$.

## Proof.

(1) $\alpha_{e n v}\left(h_{1} \bigcup h_{2}\right) \oplus \alpha_{e n v}\left(h_{1} \bigcap h_{2}\right)$

$$
\begin{aligned}
& =\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \oplus \alpha_{\text {env }}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right) \\
& =\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}, 1-\max \left\{h_{1}^{+}, h_{2}^{+}\right\}\right) \oplus\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}, 1-\min \left\{h_{1}^{+}, h_{2}^{+}\right\}\right)
\end{aligned}
$$

$$
=\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}, \min \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right) \oplus\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}, \max \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right)
$$

$$
=\left(\dot{s}^{-1}\left(\dot{s}\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}\right)+\dot{s}\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}\right)\right)\right.
$$

$$
\dot{\tau}^{-1}\left(\dot{\tau}\left(\min \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right)+\dot{\tau}\left(\max \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right)\right)
$$

$$
\begin{aligned}
&=\left(\dot{s}^{-1}\left(\max \left\{\dot{s}\left(h_{1}^{-}\right), \dot{s}\left(h_{2}^{-}\right)\right\}+\min \left\{\dot{s}\left(h_{1}^{-}\right), \dot{s}\left(h_{2}^{-}\right)\right\}\right),\right. \\
&\left.\dot{\tau}^{-1}\left(\min \left\{\dot{\tau}\left(1-h_{1}^{+}\right), \dot{\tau}\left(1-h_{2}^{+}\right)\right\}+\max \left\{\dot{\tau}\left(1-h_{1}^{+}\right), \dot{\tau}\left(1-h_{2}^{+}\right)\right\}\right)\right) \\
&=\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{2}^{-}\right)\right), \dot{\tau}^{-1}\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{2}^{+}\right)\right)\right) \\
&=\left(\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{2}^{-}\right)\right), 1-\dot{s}^{-1}\left(\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{2}^{+}\right)\right)\right)=\alpha_{e n v}\left(h_{1} \oplus h_{2}\right) . \\
& \text { (2) } \alpha_{e n v}\left(h_{1} \cup h_{2}\right) \otimes \alpha_{e n v}\left(h_{1} \cap h_{2}\right) \\
&= \alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right) \otimes \alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \min \left\{\gamma_{1}, \gamma_{2}\right\}\right) \\
&=\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}, 1-\max \left\{h_{1}^{+}, h_{2}^{+}\right\}\right) \otimes\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}, 1-\min \left\{h_{1}^{+}, h_{2}^{+}\right\}\right) \\
&=\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}, \min \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right) \otimes\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}, \max \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right) \\
&=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(\max \left\{h_{1}^{-}, h_{2}^{-}\right\}\right)+\dot{\tau}\left(\min \left\{h_{1}^{-}, h_{2}^{-}\right\}\right)\right),\right. \\
&\left.\dot{s}^{-1}\left(\dot{s}\left(\min \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right)+\dot{s}\left(\max \left\{1-h_{1}^{+}, 1-h_{2}^{+}\right\}\right)\right)\right) \\
&=\left(\dot{\tau}^{-1}\left(\max \left\{\dot{\tau}\left(h_{1}^{-}\right), \dot{\tau}\left(h_{2}^{-}\right)\right\}+\min \left\{\dot{\tau}\left(h_{1}^{-}\right), \dot{\tau}\left(h_{2}^{-}\right)\right\}\right),\right. \\
&\left.\dot{s}^{-1}\left(\min \left\{\dot{s}\left(1-h_{1}^{+}\right), \dot{s}\left(1-h_{2}^{+}\right)\right\}+\max \left\{\dot{s}\left(1-h_{1}^{+}\right), \dot{s}\left(1-h_{2}^{+}\right)\right\}\right)\right) \\
&=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{2}^{-}\right)\right), \dot{s}^{-1}\left(\dot{s}\left(1-h_{1}^{+}\right)+\dot{s}\left(1-h_{2}^{+}\right)\right)\right) \\
&=\left(\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{-}\right)+\dot{\tau}\left(h_{2}^{-}\right)\right), 1-\dot{\tau}^{-1}\left(\dot{\tau}\left(h_{1}^{+}\right)+\dot{\tau}\left(h_{2}^{+}\right)\right)\right)=\alpha_{e n v}\left(h_{1} \otimes h_{2}\right) .
\end{aligned}
$$

Theorem 1.19 (Xia and Xu 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, and $\lambda>0$, then
(1) $\lambda\left(h_{1} \cup h_{2}\right)=\lambda h_{1} \cup \lambda h_{2}$.
(2) $\lambda\left(h_{1} \cap h_{2}\right)=\lambda h_{1} \cap \lambda h_{2}$.
(3) $\left(h_{1} \cup h_{2}\right)^{\lambda}=h_{1}^{\lambda} \cup h_{2}^{\lambda}$.
(4) $\left(h_{1} \cap h_{2}\right)^{\lambda}=h_{1}^{\lambda} \cap h_{2}^{\lambda}$.
(5) $\lambda\left(h_{1} \oplus h_{2}\right)=\lambda h_{1} \oplus \lambda h_{2}$.
(6) $\left(h_{1} \otimes h_{2}\right)^{\lambda}=h_{1}^{\lambda} \otimes h_{2}^{\lambda}$.

Proof. In the following, we prove (1), (3) and (5), others can be proven analogously:
(1) $\lambda\left(h_{1} \cup h_{2}\right)=\lambda\left(\bigcup_{\gamma \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right)=\dot{s}^{-1}\left(\lambda \dot{s}\left(\bigcup_{\gamma \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right)\right)$

$$
=\underset{\gamma \in h_{1}, \gamma_{s} l_{2}}{ } \max \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(\gamma_{1}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(\gamma_{2}\right)\right)\right\}=\lambda h_{1} \cup \lambda h_{2} .
$$

(3) $\left(h_{1} \cup h_{2}\right)^{\lambda}=\left(\bigcup_{\gamma \in h_{1}, \gamma_{2} \leqslant h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right)^{\lambda}=i^{-1}\left(\lambda i\left(\bigcup_{\gamma \in h_{1}, v_{E} \in h_{2}} \max \left\{\gamma_{1}, \gamma_{2}\right\}\right)\right)$

$$
=\bigcup_{\gamma_{\in} \in h_{1}, z_{2} \xi_{2}} \max \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{1}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{2}\right)\right)\right\}=h_{1}^{\lambda} \cup h_{2}^{\lambda} .
$$

(5) $\lambda\left(h_{1} \oplus h_{2}\right)=\lambda\left(\bigcup_{\gamma \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right\}\right)$.
$=\underset{\gamma_{1} \in h_{1}, \gamma_{2} \xi_{2}}{ }\left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right)\right\}\right.$
$=\bigcup_{\gamma_{1} \in \epsilon_{1}, \gamma_{2} \xi_{2}}\left\{\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right)\right\}$
$=\underset{\gamma_{1} \in h_{1}, \gamma_{2} b_{2}}{ }\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(\gamma_{1}\right)\right)\right)+\dot{s}\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(\gamma_{2}\right)\right)\right)\right)\right)\right\}\right.$
$=\lambda h_{1} \oplus \lambda h_{2}$.

Theorem 1.20 (Xia and Xu 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $\alpha_{e n v}\left(\lambda\left(h_{1} \cup h_{2}\right)\right)=\alpha_{e n v}\left(\lambda h_{1}\right) \cup \alpha_{e n v}\left(\lambda h_{2}\right)$.
(2) $\alpha_{e n v}\left(\lambda\left(h_{1} \bigcap h_{2}\right)\right)=\alpha_{e n v}\left(\lambda h_{1}\right) \bigcap \alpha_{e n v}\left(\lambda h_{2}\right)$.
(3) $\alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right)^{\lambda}\right)=\alpha_{e n v}\left(h_{1}^{\lambda}\right) \cup \alpha_{e n v}\left(h_{2}^{\lambda}\right)$.
(4) $\alpha_{e n v}\left(\left(h_{1} \cap h_{2}\right)^{\lambda}\right)=\alpha_{e n v}\left(h_{1}^{\lambda}\right) \cap \alpha_{e n v}\left(h_{2}^{\lambda}\right)$.
(5) $\alpha_{e n v}\left(\lambda\left(h_{1} \oplus h_{2}\right)\right)=\alpha_{e n v}\left(\lambda h_{1}\right) \oplus \alpha_{e n v}\left(\lambda h_{2}\right)$.
(6) $\alpha_{e n v}\left(\left(h_{1} \otimes h_{2}\right)^{\lambda}\right)=\alpha_{e n v}\left(h_{1}^{\lambda}\right) \otimes \alpha_{e n v}\left(h_{2}^{\lambda}\right)$.

Proof. In the following, we prove (1), (3) and (5), others can be proven analogously:

$$
\text { (1) } \alpha_{e n v}\left(\lambda\left(h_{1} \bigcup h_{2}\right)\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(\gamma_{1}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(\gamma_{2}\right)\right)\right\}\right)
$$

$$
\begin{aligned}
& =\left(\max \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{-}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{-}\right)\right)\right\}, 1-\max \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{+}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{+}\right)\right)\right\}\right) \\
& =\left(\max \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{-}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{-}\right)\right)\right\}, \min \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(1-h_{1}^{+}\right)\right), i^{-1}\left(\lambda \dot{\tau}\left(1-h_{2}^{+}\right)\right)\right\}\right) \\
& =\left(\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{-}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(1-h_{1}^{+}\right)\right)\right) \cup\left(\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{-}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(1-h_{2}^{+}\right)\right)\right) \\
& =\left(\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{-}\right)\right), 1-\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{1}^{+}\right)\right)\right) \cup\left(\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{-}\right)\right), 1-\dot{s}^{-1}\left(\lambda \dot{s}\left(h_{2}^{+}\right)\right)\right) \\
& =\alpha_{e n v}\left(\lambda h_{1}\right) \cup \alpha_{e n v}\left(\lambda h_{2}\right) .
\end{aligned}
$$

$$
\text { (3) } \alpha_{e n v}\left(\left(h_{1} \cup h_{2}\right)^{\lambda}\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}} \max \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{1}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{2}\right)\right)\right\}\right)
$$

$$
=\left(\max \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{-}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{-}\right)\right)\right\}, 1-\max \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{+}\right)\right), \dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{+}\right)\right)\right\}\right)
$$

$$
=\left(\max \left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{-}\right)\right), i^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{-}\right)\right)\right\}, \min \left\{\dot{s}^{-1}\left(\lambda \dot{s}\left(1-h_{1}^{+}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(1-h_{2}^{+}\right)\right)\right\}\right)
$$

$$
=\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{-}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(1-h_{1}^{+}\right)\right)\right) \cup\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{-}\right)\right), \dot{s}^{-1}\left(\lambda \dot{s}\left(1-h_{2}^{+}\right)\right)\right)
$$

$$
=\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{-}\right), 1-\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{1}^{+}\right)\right)\right) \cup\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{-}\right)\right), 1-\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(h_{2}^{+}\right)\right)\right) .\right.
$$

$$
=\alpha_{e n v}\left(h_{1}^{\lambda}\right) \cup \alpha_{e n v}\left(h_{2}^{\lambda}\right)
$$

$$
\text { (5) } \begin{aligned}
& \alpha_{e n v}\left(\lambda\left(h_{1} \oplus h_{2}\right)\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(\gamma_{1}\right)+\dot{s}\left(\gamma_{2}\right)\right)\right)\right\}\right) \\
&=\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{2}^{-}\right)\right)\right), 1-\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{+}\right)+\dot{s}\left(h_{2}^{+}\right)\right)\right)\right) \\
&=\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{-}\right)+\dot{s}\left(h_{2}^{-}\right)\right)\right), \dot{\tau}^{-1}\left(\lambda\left(\dot{\tau}\left(1-h_{1}^{+}\right)+\dot{\tau}\left(1-h_{2}^{+}\right)\right)\right)\right) \\
&=\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{-}\right)\right)\right), \dot{\tau}^{-1}\left(\lambda\left(\dot{\tau}\left(1-h_{1}^{+}\right)\right)\right)\right)+\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{2}^{-}\right)\right)\right), \dot{\tau}^{-1}\left(\lambda\left(\dot{\tau}\left(1-h_{2}^{+}\right)\right)\right)\right) \\
&=\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{-}\right)\right)\right), 1-\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{1}^{+}\right)\right)\right)\right)+\left(\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{2}^{-}\right)\right)\right), 1-\dot{s}^{-1}\left(\lambda\left(\dot{s}\left(h_{2}^{+}\right)\right)\right)\right) \\
&=\alpha_{e n v}\left(\lambda h_{1}\right) \oplus \alpha_{e n v}\left(\lambda h_{2}\right) .
\end{aligned}
$$

### 1.2 Hesitant Fuzzy Aggregation Operators

In the previous section, we have revealed the relations between HFEs and IFNs. Recently, lots of operators have been developed for aggregating IFNs, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, the intuitionistic fuzzy hybrid averaging (IFHA) operator, the
intuitionistic fuzzy hybrid geometric (IFHG) operator, and the generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator, etc. In the following, we first review some common intuitionistic fuzzy aggregation operators:

For a collection of IFNs $\alpha_{i}=\left(\mu_{i}, v_{i}\right)(i=1,2, \cdots, n)$, where $\mu_{i}, v_{i} \in[0,1]$, and $\mu_{i}+v_{i} \leq 1, i=1,2, \cdots, n$, then
(1) The intuitionistic fuzzy weighted averaging (IFWA) operator (Xu 2007a):

$$
\begin{equation*}
\operatorname{IFWA}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\oplus_{i=1}^{n} w_{i} \alpha_{i}=\left(1-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{i}}\right)^{w_{i}}, \prod_{i=1}^{n} v_{\alpha_{i}}^{w_{i}}\right) \tag{1.24}
\end{equation*}
$$

where $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$.
(2) The intuitionistic fuzzy weighted geometric (IFWG) operator (Xu and Yager 2006):

$$
\begin{equation*}
\operatorname{IFWG}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\bigotimes_{i=1}^{n} \alpha_{i}^{w_{i}}=\left(\prod_{i=1}^{n} \mu_{\alpha_{i}}^{w_{i}}, 1-\prod_{i=1}^{n}\left(1-v_{\alpha_{i}}\right)^{w_{i}}\right) \tag{1.25}
\end{equation*}
$$

where $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$.
(3) The intuitionistic fuzzy ordered weighted averaging (IFOWA) operator (Xu 2007a):
$\operatorname{IFOWA}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\oplus_{i=1}^{n} \omega_{i} \alpha_{\sigma(i)}=\left(1-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}\right)^{\omega_{i}}, \prod_{i=1}^{n} v_{\alpha_{\sigma(i)}}^{\omega_{i}}\right)$
where $\alpha_{\sigma(i)}$ is the $i$ th largest of $\alpha_{i}(i=1,2, \cdots, n)$, and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$.
(4) The intuitionistic fuzzy ordered weighted geometric (IFOWG) operator (Xu and Yager 2006):

$$
\begin{equation*}
\operatorname{IFOWG}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\oplus_{i=1}^{n} \alpha_{\sigma(i)}^{\omega_{i}}=\left(\prod_{i=1}^{n} \mu_{\alpha_{\sigma(i)}}^{\omega_{i}}, 1-\prod_{i=1}^{n}\left(1-v_{\alpha_{\sigma(i)}}\right)^{\omega_{i}}\right) \tag{1.27}
\end{equation*}
$$

where $\alpha_{\sigma(i)}$ is the $i$ th largest of $\alpha_{i}(i=1,2, \cdots, n)$, and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$.
(5) The intuitionistic fuzzy hybrid averaging (IFHA) operator (Xu 2007a):

$$
\begin{equation*}
\operatorname{IFHA}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\oplus_{i=1}^{n} \omega_{i} \dot{\alpha}_{\sigma(i)}=\left(1-\prod_{i=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(i)}}\right)^{\omega_{i}}, \prod_{i=1}^{n} v_{\dot{\alpha}_{\sigma(i)}}^{\omega_{i}}\right) \tag{1.28}
\end{equation*}
$$

where $\quad \dot{\alpha}_{\sigma(i)} \quad$ is the $i$ th largest of $\quad \dot{\alpha}_{i}=n w_{i} \alpha_{i}(i=1,2, \cdots, n)$, $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ with $w_{i} \in[0,1]$, $i=1,2, \cdots, n, \quad \sum_{i=1}^{n} w_{i}=1$, and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the aggregationassociated vector such that $\omega_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$.
(6) The intuitionistic fuzzy hybrid geometric (IFHG) operator (Xu and Yager 2006):

$$
\begin{equation*}
\operatorname{IFHG}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\oplus_{i=1}^{n} \ddot{\alpha}_{\sigma(i)}^{\omega_{i}}=\left(\prod_{i=1}^{n} \mu_{\ddot{\alpha}_{\sigma(i)}}^{\omega_{i}}, 1-\prod_{i=1}^{n}\left(1-v_{\ddot{\alpha}_{\sigma(i)}}\right)^{\omega_{i}}\right) \tag{1.29}
\end{equation*}
$$

where $\quad \ddot{\alpha}_{\sigma(i)} \quad$ is the $i$ th largest of $\quad \ddot{\alpha}_{i}=\alpha_{i}^{n w_{i}}(i=1,2, \cdots, n)$, $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}} \quad$ is the weight vector of $\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)$ with $w_{i} \in[0,1], \quad i=1,2, \cdots, n, \quad \sum_{i=1}^{n} w_{i}=1, \quad$ and $\quad \omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$.

Yager (2004a) defined a generalized ordered weighted averaging (GOWA) operator, Zhao et al. (2010) extended it to accommodate the situations where the input arguments are IFNs.

Definition 1.14 (Zhao et al. 2010). A generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator of dimension $n$ has the following form:

$$
\begin{align*}
& \operatorname{GIFOWA}\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)=\left(\oplus_{i=1}^{n} \omega_{i} \alpha_{\sigma(i)}^{\lambda}\right)^{\frac{1}{\lambda}} \\
& \quad=\left(\left(1-\prod_{i=1}^{n}\left(1-\mu_{\alpha_{\sigma(i)}}^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-v_{\alpha_{\sigma(i)}}\right)^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}\right) \tag{1.30}
\end{align*}
$$

where $\lambda>0, \omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{\mathrm{T}}$ is the weighting vector associated with the GIFOWA operator with $\omega_{i} \in[0,1], i=1,2, \ldots, n$ and $\sum_{i=1}^{n} \omega_{i}=1$, and $\alpha_{\sigma(i)}$ is the $i$ th largest of $\alpha_{i}(i=1,2, \ldots, n)$.

Furthermore, Torra and Narukawa (2009) proposed an aggregation principle for HFEs:

Definition 1.15 (Torra and Narukawa 2009). Let $A=\left\{h_{1}, h_{2}, \cdots, h_{n}\right\}$ be a set of $n$ HFEs, $\vartheta$ a function on $A, \vartheta:[0,1]^{n} \rightarrow[0,1]$, then

$$
\begin{equation*}
\vartheta_{A}=\bigcup_{\gamma \in\left\{h_{1} \times h_{2} \times \cdots \times h_{n}\right\}}\{\vartheta(\gamma)\} \tag{1.31}
\end{equation*}
$$

Based on Definition 1.15 and the given operations for HFEs, below we will introduce a series of specific aggregation operators for HFEs, and investigate their desirable properties:

Definition 1.16 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs. A hesitant fuzzy weighted averaging (HFWA) operator is a mapping $\Theta^{n} \rightarrow \Theta$ such that

$$
\begin{equation*}
\operatorname{HFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\oplus_{i=1}^{n} w_{i} h_{i}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}}\right\} \tag{1.32}
\end{equation*}
$$

where $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $h_{i}(i=1,2, \cdots, n)$ with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$. Especially, if $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the HFWA operator reduces to the hesitant fuzzy averaging (HFA) operator:

$$
\begin{equation*}
\operatorname{HFA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{1}{n} \oplus_{i=1}^{n} h_{i}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{\frac{1}{n}}\right\} \tag{1.33}
\end{equation*}
$$

Definition 1.17 (Xia and Xu 2011a). Let $h_{j}(j=1,2, \cdots, n)$ be a collection of HFEs and let HFWG: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{HFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigotimes_{i=1}^{n} h_{i}^{w_{i}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n} \gamma_{i}^{w_{i}}\right\} \tag{1.34}
\end{equation*}
$$

then HFWG is called a hesitant fuzzy weighted geometric (HFWG) operator, where $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $h_{i}(i=1,2, \cdots, n)$ with $w_{i} \in[0,1], \quad i=1,2, \cdots, n, \quad$ and $\sum_{i=1}^{n} w_{i}=1$. In the case where $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, the HFWA operator reduces to the hesitant fuzzy geometric (HFG) operator:

$$
\begin{equation*}
\operatorname{HFG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\otimes_{i=1}^{n} h_{i}^{\frac{1}{n}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n} \gamma_{i}^{\frac{1}{n}}\right\} \tag{1.35}
\end{equation*}
$$

Lemma 1.1 (Xu 2000; Torra and Narukawa 2007). Let $x_{i}>0, \lambda_{i}>0, i=1,2$, $\cdots, n$, and $\sum_{i=1}^{n} \lambda_{i}=1$, then

$$
\begin{equation*}
\prod_{i=1}^{n} x_{i}^{\lambda_{i}} \leq \sum_{i=1}^{n} \lambda_{i} x_{i} \tag{1.36}
\end{equation*}
$$

with equality if and only if $x_{1}=x_{2}=\cdots=x_{n}$.

Theorem 1.21 (Xia and Xu 2011a). Assume that $h_{i}(i=1,2, \cdots, n)$ is a collection of HFEs, whose weight vector is $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$, with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$, then

$$
\begin{equation*}
\operatorname{HFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \leq \operatorname{HFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.37}
\end{equation*}
$$

Proof. For any $\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}$, based on Lemma 1.1, we have

$$
\begin{equation*}
\prod_{i=1}^{n} \gamma_{i}^{w_{i}} \leq \sum_{i=1}^{n} w_{i} \gamma_{i}=1-\sum_{i=1}^{n} w_{i}\left(1-\gamma_{i}\right) \leq 1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}} \tag{1.38}
\end{equation*}
$$

which implies that ${\underset{i=1}{n}}_{\otimes_{i}}^{w_{i}} \leq \bigoplus_{i=1}^{n} w_{i} h_{i}$.
Theorem 1.21 shows that the values obtained by the HFWG operator are not bigger than the ones obtained by the HFWA operator.

Definition 1.18 (Xia and Xu 2011a). For a collection of the HFEs $h_{i}(i=1,2, \cdots, n)$, a generalized hesitant fuzzy weighted averaging (GHFWA) operator is a mapping GHFWA: $\Theta^{n} \rightarrow \Theta$ such that

GHFWA $_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\left(\oplus_{i=1}^{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\} \tag{1.39}
\end{equation*}
$$

where $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ is the weight vector of $h_{i}(i=1,2, \cdots, n)$, with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$. Especially, if $\lambda=1$, then the GHFWA operator reduces to the HFWA operator; If $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the GHFWA operator reduces to the GHFA operator .

Theorem 1.22 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs having the weight vector $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1]$, $i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, \lambda>0$, then

$$
\begin{equation*}
\operatorname{HFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \leq \operatorname{GHFWA}_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.40}
\end{equation*}
$$

Proof. For any $\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}$, based on Lemma 1.1, we have

$$
\begin{align*}
\prod_{i=1}^{n} \gamma_{i}^{w_{i}}= & \left(\prod_{i=1}^{n}\left(\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}} \leq\left(\sum_{i=1}^{n} w_{i} \gamma_{i}^{\lambda}\right)^{\frac{1}{\lambda}} \\
& =\left(1-\sum_{i=1}^{n} w_{i}\left(1-\gamma_{i}^{\lambda}\right)\right)^{\frac{1}{\lambda}} \leq\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}} \tag{1.41}
\end{align*}
$$

which implies that $\bigotimes_{i=1}^{n} h_{i}^{w_{i}} \leq\left(\bigoplus_{i=1}^{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}}$, and completes the proof of the theorem.

From Theorem 1.21, we can conclude that the values obtained by the HFWG operator are not bigger than the ones obtained by the GHFWA operator for any $\lambda>0$.

Definition 1.19 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs having the weight vector $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1]$, $i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, \lambda>0$. A generalized hesitant fuzzy weighted geometric (GHFWG) operator is a mapping $\Theta^{n} \rightarrow \Theta$, and

$$
\begin{align*}
& \text { GHFWG }_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{1}{\lambda}\left(\underset{i=1}{\otimes}\left(\lambda h_{i}\right)^{w_{i}}\right) \\
&=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\} \tag{1.42}
\end{align*}
$$

Especially, if $\lambda=1$, then the GHFWG operator becomes the HFWG operator; If $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the GHFWG operator reduces to the GHFG operator.

Theorem 1.23 (Xia and Xu 2011a). For a collection of HFEs $h_{i}(i=1,2, \cdots, n)$, $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}} \quad$ is the weight vector such that $w_{i} \in[0,1]$, $i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, \lambda>0$, then

$$
\begin{equation*}
\operatorname{GHFWG}_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \leq \operatorname{HFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.43}
\end{equation*}
$$

Proof. Let $\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}$, based on Lemma 1.1, we can obtain

$$
\begin{align*}
& 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}} \leq 1-\left(1-\sum_{i=1}^{n} w_{i}\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}} \\
& \quad=1-\left(\sum_{i=1}^{n} w_{i}\left(1-\gamma_{i}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq 1-\left(\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i} \lambda}\right)^{\frac{1}{\lambda}}=1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}} \tag{1.44}
\end{align*}
$$

which implies that $\frac{1}{\lambda}\left(\underset{i=1}{\otimes}\left(\lambda h_{i}\right)^{w_{i}}\right) \leq \bigoplus_{i=1}^{n} w_{i} h_{i}$, and completes the proof of the theorem.

Theorem 1.22 gives us the result that the values obtained by the GHFWG operator are not bigger than the ones obtained by the HFWA operator, no matter how the parameter $\lambda(\lambda>0)$ changes.

Example 1.6 (Xia and Xu 2011a). Let $h_{1}=\{0.2,0.3,0.5\}$ and $h_{2}=\{0.4,0.6\}$ be two HFEs, and $w=(0.7,0.3)^{\mathrm{T}}$ their weight vector, then we have

GHFWA $_{1}\left(h_{1}, h_{2}\right)=$ HFWA

$$
\left(h_{1}, h_{2}\right)=\oplus_{i=1}^{2}\left(\sum_{i=1}^{2} w_{i} h_{i}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\prod_{i=1}^{2}\left(1-\gamma_{i}\right)^{w_{i}}\right\}
$$

$$
\begin{aligned}
& =\left\{1-(1-0.2)^{0.7} \times(1-0.4)^{0.3}, 1-(1-0.2)^{0.7} \times(1-0.6)^{0.3}, 1-(1-0.3)^{0.7} \times(1-0.4)^{0.3},\right. \\
& \left.1-(1-0.3)^{0.7} \times(1-0.6)^{0.3}, 1-(1-0.5)^{0.7} \times(1-0.4)^{0.3}, 1-(1-0.5)^{0.7} \times(1-0.6)^{0.3}\right\} \\
& =\{0.2661,0.3316,0.3502,0.4082,0.4719,0.5324\}
\end{aligned}
$$

$$
\begin{aligned}
& \operatorname{GHFWA}_{6}\left(h_{1}, h_{2}\right)=\left({ \underset { i = 1 } { 2 } w _ { i } h _ { i } ^ { 6 } ) ^ { \frac { 1 } { 6 } } = \bigcup _ { \gamma _ { 1 } \in h _ { 1 } , \gamma _ { 2 } \in h _ { 2 } } \{ ( 1 - \prod _ { i = 1 } ^ { 2 } ( 1 - \gamma _ { i } ^ { 6 } ) ^ { w _ { i } } ) ^ { \frac { 1 } { 6 } } \} } _ { = } \left\{\left(1-\left(1-0.2^{6}\right)^{0.7} \times\left(1-0.4^{6}\right)^{0.3}\right)^{\frac{1}{6}},\left(1-\left(1-0.2^{6}\right)^{0.7} \times\left(1-0.6^{6}\right)^{0.3}\right)^{\frac{1}{6}},\right.\right. \\
&\left(1-\left(1-0.3^{6}\right)^{0.7} \times\left(1-0.4^{6}\right)^{0.3}\right)^{\frac{1}{6}}\left(1-\left(1-0.3^{6}\right)^{0.7} \times\left(1-0.6^{6}\right)^{0.3}\right)^{\frac{1}{6}}, \\
&\left.\left(1-\left(1-0.5^{6}\right)^{0.7} \times\left(1-0.4^{6}\right)^{0.3}\right)^{\frac{1}{6}},\left(1-\left(1-0.5^{6}\right)^{0.7} \times\left(1-0.6^{6}\right)^{0.3}\right)^{\frac{1}{6}}\right\}
\end{aligned}
$$

$$
=\{0.3293,0.3468,0.4707,0.4925,0.4951,0.5409\}
$$

$$
\begin{aligned}
& \operatorname{GHFWG}_{1}\left(h_{1}, h_{2}\right)=\operatorname{HFWG}\left(h_{1}, h_{2}\right)=\oplus_{i=1}^{2} h_{i}^{w_{i}}=\bigcup_{\gamma_{1} h_{1}, \gamma_{2} \in h_{2}}\left\{\prod_{i=1}^{2} \gamma_{i}^{w_{i}}\right\} \\
&=\left\{0.2^{0.7} \times 0.4^{0.3}, 0.2^{0.7} \times 0.6^{0.3}, 0.3^{0.7} \times 0.4^{0.3}, 0.3^{0.7} \times 0.6^{0.3},\right. \\
&\left.0.5^{0.7} \times 0.4^{0.3}, 0.5^{0.7} \times 0.6^{0.3}\right\} \\
&=\{0.2462,0.2781,0.3270,0.3693,0.4676,0.5281\}
\end{aligned}
$$

$\operatorname{GHFWG}_{6}\left(h_{1}, h_{2}\right)=\frac{1}{6}\left(\underset{i=1}{\otimes}\left(6 h_{i}\right)^{w_{i}}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{1-\left(1-\prod_{i=1}^{2}\left(1-\left(1-\gamma_{i}\right)^{6}\right)^{w_{i}}\right)^{\frac{1}{6}}\right\}$

$$
\begin{aligned}
= & \left\{1-\left(1-\left(1-(1-0.2)^{6}\right)^{0.7} \times\left(1-(1-0.4)^{6}\right)^{0.3}\right)^{\frac{1}{6}},\right. \\
& 1-\left(1-\left(1-(1-0.2)^{6}\right)^{0.7} \times\left(1-(1-0.6)^{6}\right)^{0.3}\right)^{\frac{1}{6}}, \\
& 1-\left(1-\left(1-(1-0.3)^{6}\right)^{0.7} \times\left(1-(1-0.4)^{6}\right)^{0.3}\right)^{\frac{1}{6}}, \\
& 1-\left(1-\left(1-(1-0.3)^{6}\right)^{0.7} \times\left(1-(1-0.6)^{6}\right)^{0.3}\right)^{\frac{1}{6}}, \\
& 1-\left(1-\left(1-(1-0.5)^{6}\right)^{0.7} \times\left(1-(1-0.4)^{6}\right)^{0.3}\right)^{\frac{1}{6}}, \\
& \left.1-\left(1-\left(1-(1-0.5)^{6}\right)^{0.7} \times\left(1-(1-0.6)^{6}\right)^{0.3}\right)^{\frac{1}{6}}\right\} \\
= & \{0.2333,0.2400,0.3222,0.3369,0.4591,0.5203\}
\end{aligned}
$$

In the following, we discuss the relationships among the developed aggregation operators:

Theorem 1.24 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1]$, $i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, \lambda>0$, then
(1) $\oplus_{i=1}^{n} w_{i} h_{i}^{c}=\left(\stackrel{n}{\otimes}_{i=1}^{n} h_{i}^{w_{i}}\right)^{c}$.
(2) $\stackrel{n}{\otimes}\left(h_{i=1}^{c}\right)^{w_{i}}=\left(\bigoplus_{i=1}^{n} w_{i} h_{i}\right)^{c}$.
(3) $\left(\bigoplus_{i=1}^{n} w_{i}\left(h_{i}^{c}\right)^{\lambda}\right)^{\frac{1}{\lambda}}=\left(\frac{1}{\lambda}\left(\underset{i=1}{n}\left(\lambda h_{i}\right)^{w_{i}}\right)\right)^{c}$.

Proof. (1) $\oplus_{i=1}^{n}\left(w_{i} h_{i}^{c}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{i=1}^{n}\left(\gamma_{i}\right)^{w_{i}}\right\}$

$$
=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\gamma_{i}\right)^{w_{i}}\right\}\right)^{c}=\left(\otimes_{i=1}^{n} h_{i}^{w_{i}}\right)^{c} .
$$

(2) $\bigotimes_{i=1}^{\otimes}\left(h_{i}^{c}\right)^{w_{i}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}}\right\}$

$$
=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}}\right\}\right)^{c}=\left({ }_{i=1}^{n} w_{i} h_{i}\right)^{c} .
$$

(3) $\left(\underset{i=1}{n}\left(w_{i}\left(h_{i}^{c}\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}$

$$
=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right)^{c}=\left(\frac{1}{\lambda}\left(\bigotimes_{i=1}^{\otimes}\left(\lambda h_{i}\right)^{w_{i}}\right)\right)^{c}
$$

(4) $\frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}\left(\lambda h_{i}^{c}\right)^{w_{i}}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}$

$$
=\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right)^{c}=\left(\oplus_{i=1}^{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}} .
$$

Theorem 1.25 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs associated with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, \lambda>0$, then
(1) $\alpha_{e n v}\left(\oplus_{i=1}^{n} w_{i} h_{i}\right)=\bigoplus_{i=1}^{n} w_{i} \alpha_{e n v}\left(h_{i}\right)$.
(2) $\alpha_{e n v}\left(\stackrel{\bigotimes}{\otimes}_{i=1}^{n} w_{i} h_{i}\right)=\bigotimes_{i=1}^{n} w_{i} \alpha_{e n v}\left(h_{i}\right)$.
(3) $\alpha_{e n v}\left(\left(\oplus_{i=1}^{n} w_{i}\left(h_{i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)=\left(\oplus_{i=1}^{n} w_{i}\left(\alpha_{e n v}\left(h_{i}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}$.

Proof. (1) $\alpha_{e n v}\left(\oplus_{i=1}^{n} w_{i} h_{i}\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_{i}\right)^{w_{i}}\right\}\right)$

$$
=\left(1-\prod_{i=1}^{n}\left(1-h_{i}^{-}\right)^{w_{i}}, 1-\left(1-\prod_{i=1}^{n}\left(1-h_{i}^{+}\right)^{w_{i}}\right)\right)
$$

$$
=\left(1-\prod_{i=1}^{n}\left(1-h_{i}^{-}\right)^{w_{i}}, \prod_{i=1}^{n}\left(1-h_{i}^{+}\right)^{w_{i}}\right)
$$

$$
=\oplus_{i=1}^{n}\left(w_{i}\left(h_{i}^{-}, 1-h_{i}^{+}\right)\right)=\oplus_{i=1}^{n} w_{i} \alpha_{e n v}\left(h_{i}\right) .
$$

(2) $\alpha_{e n v}\left(\stackrel{n}{\otimes} h_{i=1}^{w_{i}}\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n} \gamma_{i}^{w_{i}}\right\}\right)$

$$
\begin{aligned}
& =\left(\prod_{i=1}^{n}\left(h_{i}^{-}\right)^{w_{i}}, 1-\prod_{i=1}^{n}\left(h_{i}^{+}\right)^{w_{i}}\right) \\
& =\left(\prod_{i=1}^{n}\left(h_{i}^{-}\right)^{w_{i}}, 1-\prod_{i=1}^{n}\left(1-\left(1-h_{i}^{+}\right)\right)^{w_{i}}\right) \\
& =\stackrel{n}{\otimes}\left(h_{i}^{-}, 1-h_{i}^{+}\right)^{w_{i}}=\stackrel{n}{\otimes=1}\left(\alpha_{e n v}\left(h_{i}\right)\right)^{w_{i}} .
\end{aligned}
$$

(3) $\alpha_{e n v}\left(\left(\oplus_{i=1}^{n} w_{i}\left(h_{i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{\in} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right)$

$$
=\left(\left(1-\prod_{i=1}^{n}\left(1-\left(h_{i}^{-}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}\left(1-\left(h_{i}^{+}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right) .
$$

$$
=\left(\left(1-\prod_{i=1}^{n}\left(1-\left(h_{i}^{-}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(1-h_{i}^{+}\right)\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right)
$$

$$
=\left({ }_{i=1}^{\oplus} w_{i}\left(h_{i}^{-}, 1-h_{i}^{+}\right)^{\lambda}\right)^{\frac{1}{\lambda}}=\left({ }_{i=1}^{\oplus} w_{i}\left(\alpha_{e n v}\left(h_{i}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} .
$$

(4) $\alpha_{e n v}\left(\frac{1}{\lambda}\left(\left(_{i=1}^{n}\left(\lambda h_{i}\right)^{w_{i}}\right)\right)=\alpha_{e n v}\left(\bigcup_{\gamma_{n} \in h_{1}, \gamma_{2} \in h_{2}, \cdots \gamma_{n} \in h_{n}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{i}\right)^{2}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right)\right.$

$$
=\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-h_{i}^{-}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}},\left(1-\prod_{i=1}^{n}\left(1-\left(1-h_{i}^{+}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right)
$$

$$
=\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(\lambda\left(h_{i}^{-}, 1-h_{i}^{+}\right)\right)^{w_{i}}\right)=\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(\lambda \alpha_{e n v}\left(h_{i}\right)\right)^{w_{i}}\right) .
$$

Definition 1.20 (Xia and Xu 2011a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs, $h_{\sigma(i)}$ the $i$ th largest of them, $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ the aggregationassociated vector such that $\omega_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$, then
(1) A hesitant fuzzy ordered weighted averaging (HFOWA) operator is a mapping HFOWA: $\Theta^{n} \rightarrow \Theta$, such that
$\operatorname{HFOWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\oplus_{i=1}^{n}\left(\sum_{i=1}^{n} \omega_{i} h_{\sigma(i)}\right)=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{\omega_{i}}\right\} \tag{1.45}
\end{equation*}
$$

(2) A hesitant fuzzy ordered weighted geometric (HFOWG) operator is a mapping HFOWG: $\Theta^{n} \rightarrow \Theta$, such that
$\operatorname{HFOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigotimes_{i=1}^{n} h_{\sigma(i)}^{\omega_{i}}=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\prod_{i=1}^{n} \gamma_{\sigma(i)}^{\omega_{i}}\right\} \tag{1.46}
\end{equation*}
$$

(3) A generalized hesitant fuzzy ordered weighted averaging (GHFOWA) operator is a mapping GHFOWA: $\Theta^{n} \rightarrow \Theta$, such that

GHFOWA $_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\left(\bigoplus_{i=1}^{n} \omega_{i} h_{\sigma(i)}^{\lambda}\right)^{\frac{1}{\lambda}}$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}\right\} \tag{1.47}
\end{equation*}
$$

with $\lambda>0$.
(4) A generalized hesitant fuzzy ordered weighted geometric (GHFOWG) operator is a mapping GHFOWG $\Theta^{n} \rightarrow \Theta$, such that
$\operatorname{GHFOWG}_{\lambda}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{1}{\lambda}\left(\underset{i=1}{\otimes}\left(\lambda h_{\sigma(i)}\right)^{\omega_{i}}\right)$

$$
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}\right\}
$$

with $\lambda>0$.

In the case where $\omega=\left(\frac{1}{n}, \frac{1}{n} \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, the HFOWA operator reduces to the HFA operator, and the HFOWG operator becomes the HFG operator; In the case where $\lambda=1$, the GHFOWA operator reduces to the HFOWA operator and the GHFOWG operator reduces to the HFOWG operator.

The HFOWA, HFOWG, GHFOWA, and HFOWG operators are based on the idea of the ordered weighted averaging (OWA) operator (Yager 1988; Yager and Kacprzyk 1997) and the ordered weighted geometric (OWG) operator (Xu and Da 2002a). The main characterization of the OWA operator is its reordering step. Several methods have been developed to obtain the OWA weights. Yager (1988) used the linguistic quantifiers to compute the OWA weights. O'Hagan (1988) generated the OWA weights with a predefined degree of orness by maximizing the entropy of the OWA weights. Filev and Yager (1998) obtained the OWA weights based on the exponential smoothing. Yager and Filev (1999) got the OWA weights from a collection of samples with the relevant aggregated data. Xu and Da (2002b) obtained the OWA weights under partial weight information by establishing a linear objective-programming model. Especially, based on the normal distribution (Gaussian distribution), Xu (2005a) developed a method to obtain the OWA weights, whose prominent characteristic is that it can relieve the influence of unfair arguments on the decision result by assigning low weights to those "false" or "biased'" ones.

Example 1.7 (Xia and Xu 2011a). Let $h_{1}=\{0.1,0.4\}, h_{2}=\{0.3,0.5\}$ and $h_{3}=\{0.2,0.5,0.8\}$ be three HFEs, and suppose that the aggregation-associated vector is $\omega=(0.25,0.4,0.35)^{\mathrm{T}}$, then we can calculate the scores of $h_{1}, h_{2}$ and $h_{3}$ :

$$
s\left(h_{1}\right)=\frac{0.1+0.4}{2}=0.25, s\left(h_{2}\right)=\frac{0.3+0.5}{2}=0.4
$$

$$
s\left(h_{3}\right)=\frac{0.2+0.5+0.8}{3}=0.5
$$

Since

$$
s\left(h_{3}\right)>s\left(h_{2}\right)>s\left(h_{1}\right)
$$

then

$$
\begin{gathered}
h_{\sigma(1)}=h_{3}=(0.2,0.5,0.8), h_{\sigma(2)}=h_{2}=(0.3,0.5) \\
h_{\sigma(3)}=h_{1}=(0.1,0.4)
\end{gathered}
$$

we have
$\operatorname{GHFOWA}_{1}\left(h_{1}, h_{2}, h_{3}\right)=\operatorname{HFOWA}\left(h_{1}, h_{2}, h_{3}\right)=\stackrel{3}{\oplus} \omega_{i=1} \omega_{i(i)}$

$$
\begin{aligned}
= & \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{1-\left(1-\gamma_{3}\right)^{0.25}\left(1-\gamma_{2}\right)^{0.4}\left(1-\gamma_{1}\right)^{0.35}\right\} \\
= & \{0.2097,0.2973,0.3092,0.3143,0.3858,0.3903,0.4006,0.4412, \\
& 0.4671,0.5115,0.5151,0.5762\}
\end{aligned}
$$

$\operatorname{GHFOWA}_{2}\left(h_{1}, h_{2}, h_{3}\right)=\left(\bigoplus_{i=1}^{3} \omega_{i} h_{\sigma(i)}^{2}\right)^{\frac{1}{2}}$

$$
\begin{aligned}
= & \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\left(1-\left(1-\gamma_{3}^{2}\right)^{0.25}\left(1-\gamma_{2}^{2}\right)^{0.4}\left(1-\gamma_{1}^{2}\right)^{0.35}\right)^{\frac{1}{2}}\right\} \\
= & \{0.2239,0.3213,0.3271,0.3476,0.3961,0.4123,0.4165,0.4687, \\
& 0.5067,0.5461,0.5586,0.5920\}
\end{aligned}
$$



$$
\begin{gathered}
\begin{array}{c}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\gamma_{3}^{0.25} \gamma_{2}^{0.4} \gamma_{1}^{0.35}\right\} \\
= \\
\{0.1845,0.2264,0.2321,0.2610,0.2847,0.2998,0.3202,0.3678 \\
\\
\\
\text { GHFWG } \left._{2}\left(h_{1}, h_{2}, h_{3}\right)=\frac{1}{2}\left({\underset{i=1}{\otimes}(2770,0.4240,0.4624,0.5201\}}_{3}^{\sigma(i)}\right)^{\omega_{i}}\right) \\
= \\
\gamma_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{1-\left(1-\left(1-\left(1-\gamma_{3}\right)^{2}\right)^{0.25}\left(1-\left(1-\gamma_{2}\right)^{2}\right)^{0.4}\left(1-\left(1-\gamma_{1}\right)^{2}\right)^{0.35}\right)^{\frac{1}{2}}\right\} \\
= \\
\{0.1820,0.2165,0.2238,0.2403,0.2678,0.2882,0.2972,0.3601, \\
0.3740,0.4057,0.4610,0.5047\}
\end{array}
\end{gathered}
$$

It is noted that the HFWA, HFWG, GHFWA and GHFWG operators only weight the hesitant fuzzy argument itself, but ignores the importance of the ordered position of the argument, while the HFOWA, HFOWG, GHFOWA and GHFOWG operators only weight the ordered position of each given argument, but ignore the importance of the argument. To avoid this drawback, it is necessary to introduce some hybrid aggregation operators for hesitant fuzzy arguments, which weight all the given arguments and their ordered positions.

Definition 1.21 (Xia and Xu 2011a). For a collection of HFEs $h_{i}(i=1,2, \cdots, n)$, $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is their weight vector with $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1, n$ is the balancing coefficient which plays a role of balance, then we define the following aggregation operators, which are all based on the mapping $\Theta^{n} \rightarrow \Theta$ with an aggregation-associated vector $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ such that $\omega_{i} \in[0,1]$ and $\sum_{i=1}^{n} \omega_{i}=1$ :
(1) The hesitant fuzzy hybrid averaging (HFHA) operator:
$\operatorname{HFHA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\oplus_{i=1}^{n} \omega_{i} \dot{h}_{\sigma(i)}$
where $\dot{h}_{\sigma(i)}$ is the $i$ th largest of $\dot{h}=n w_{k} h_{k}(k=1,2, \cdots, n)$.
(2) The hesitant fuzzy hybrid geometric (HFHG) operator:

$$
\begin{equation*}
\operatorname{HFHG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigotimes_{i=1}^{n} \ddot{h}_{\sigma(i)}^{\omega_{i}}=\bigcup_{\ddot{\gamma}_{\sigma(1)} \in \ddot{h}_{\sigma(1)}, \ddot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \cdots, \dot{\gamma}_{\sigma(n)} \in \ddot{h}_{\sigma(n)}}\left\{\prod_{i=1}^{n} \ddot{\gamma}_{\sigma(i)}^{\omega_{i}}\right\} \tag{1.49}
\end{equation*}
$$

where $\ddot{h}_{\sigma(i)}$ is the $i$ th largest of $\ddot{h}_{k}=h_{k}^{n w_{k}}(k=1,2, \cdots, n)$.
(3) The generalized hesitant fuzzy hybrid averaging (HFHA) operator:


$$
\begin{equation*}
={\dot{\dot{\gamma}_{\sigma(1)} \in \dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \cdots, \dot{\gamma}_{\sigma(n)} \in \dot{h}_{\sigma(n)}}}^{\bigcup^{\prime}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\dot{\gamma}_{\sigma(i)}^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}\right\} \tag{1.50}
\end{equation*}
$$

where $\lambda>0, \dot{h}_{\sigma(j)}$ is the $j$ th largest of $\dot{h}=n w_{k} h_{k}(k=1,2, \cdots, n)$.
(4) The generalized hesitant fuzzy hybrid geometric (GHFHG) operator:
$\operatorname{GHFHG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{1}{\lambda}\left({\left.\left.\underset{i=1}{n}\left(\lambda \ddot{h}_{\sigma(i)}\right)^{\omega_{i}}\right), ~\right) ~}_{\text {in }}\right)$

$$
\begin{equation*}
={\ddot{\ddot{\gamma}_{\sigma(1)}} \in \ddot{h}_{\sigma(1)}, \ddot{\gamma}_{\sigma(2)} \in \ddot{h}_{\sigma(2)}, \cdots, \ddot{\gamma}_{\sigma(n)} \in \ddot{h}_{\sigma(n)}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\ddot{\gamma}_{\sigma(i)}\right)^{\lambda}\right)^{\omega_{i}}\right)^{\frac{1}{\lambda}}\right\} \tag{1.51}
\end{equation*}
$$

where $\lambda>0, \ddot{h}_{\sigma(i)}$ is the $i$ th largest of $\ddot{h}_{k}=h_{k}^{n w_{k}}(k=1,2, \cdots, n)$.

Especially, if $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{T}$, then the HFHA operator reduces to the HFOWA operator, the HFHG operator reduces to the HFOWG operator, the GHFHA operator reduces to the GHFOWA operator, and the GHFHG operator becomes the GHFOWG operator; If $\lambda=1$, then the GHFHA operator reduces to the HFHA operator, and the GHFHG operator becomes the HFHG operator.

Example 1.8 (Xia and Xu 2011a). Let $h_{1}=\{0.2,0.4,0.5\}, h_{2}=\{0.2,0.6\}$ and $h_{3}=\{0.1,0.3,0.4\}$ be three HFEs, whose weight vector is $w=(0.15,0.3,0.55)^{\mathrm{T}}$, and the aggregation-associated vector is $\omega=(0.3,0.4,0.3)^{\mathrm{T}}$, then we can obtain

$$
\begin{gathered}
\dot{h}_{1}=\left\{1-(1-0.2)^{3 \times 0.15}, 1-(1-0.4)^{3 \times 0.15}, 1-(1-0.5)^{3 \times 0.15}\right\}=\{0.0955,0.2054,0.2680\} \\
\dot{h}_{2}=\left\{1-(1-0.2)^{3 \times 0.3}, 1-(1-0.6)^{3 \times 0.3}\right\}=\{0.1819,0.5616\} \\
\dot{h}_{3}=\left\{1-(1-0.1)^{3 \times 0.55}, 1-(1-0.3)^{3 \times 0.55}, 1-(1-0.4)^{3 \times 0.55}\right\}=\{0.1596,0.4448,0.5695\}
\end{gathered}
$$

and

$$
s\left(\dot{h}_{1}\right)=0.1896, s\left(\dot{h}_{2}\right)=0.3718, s\left(\dot{h}_{3}\right)=0.3913
$$

Since

$$
s\left(\dot{h}_{3}\right)>s\left(\dot{h}_{2}\right)>s\left(\dot{h}_{1}\right)
$$

then

$$
\begin{gathered}
\dot{h}_{\sigma(1)}=\dot{h}_{3}=\{0.1596,0.4448,0.5695\}, \dot{h}_{\sigma(2)}=\dot{h}_{2}=\{0.1819,0.5616\} \\
\dot{h}_{\sigma(3)}=\dot{h}_{1}=\{0.0955,0.2054,0.2680\}
\end{gathered}
$$

Thus, we have

$$
\begin{aligned}
& \operatorname{GHFHA}_{1}\left(h_{1}, h_{2}, h_{3}\right)=\operatorname{HFHA}\left(h_{1}, h_{2}, h_{3}\right)=\oplus_{i=1}^{3} \omega_{i} \dot{h}_{\sigma(i)} \\
& \quad=\bigcup_{\dot{\gamma}_{1} \in \dot{h}_{1}, \dot{\gamma}_{2} \in \dot{h}_{2}, \dot{\gamma}_{3} \in \dot{h}_{3}}\left\{1-\left(1-\dot{\gamma}_{3}\right)^{0.3}\left(1-\dot{\gamma}_{2}\right)^{0.4}\left(1-\dot{\gamma}_{1}\right)^{0.3}\right\}
\end{aligned}
$$

$$
=\{0.1501,0.1825,0.2023,0.2494,0.2781,0.2956,0.3046,0.3311,0.3378
$$

$0.3474,0.3630,0.3785,0.4152,0.4375,0.4512,0.4582,0.4788,0.4915\}$

$$
\begin{aligned}
& \operatorname{GHFHA}_{3}\left(h_{1}, h_{2}, h_{3}\right)=\left(\oplus_{i=1}^{3} \omega_{i} \dot{h}_{\sigma(i)}^{3}\right)^{\frac{1}{3}} \\
& \quad=\bigcup_{\dot{\gamma}_{1} \in \dot{h}_{1}, \dot{\gamma}_{2} \in \dot{h}_{2}, \dot{\gamma}_{3} \in \dot{h}_{3}}\left\{\left(1-\left(1-\dot{\gamma}_{3}^{3}\right)^{0.3}\left(1-\dot{\gamma}_{2}^{3}\right)^{0.4}\left(1-\dot{\gamma}_{1}^{3}\right)^{0.3}\right)^{\frac{1}{3}}\right\}
\end{aligned}
$$

$$
=\{0.1573,0.1840,0.2112,0.3102,0.3179,0.3279,0.3957,0.4243,0.4283,
$$

$$
0.4336,0.4649,0.4681,0.4725,0.4003,0.4065,0.5069,0.5095,0.5130\}
$$

If we use the GHFHG operator to aggregate the HFEs $h_{1}, h_{2}$ and $h_{3}$, then

$$
\begin{gathered}
\ddot{h}_{\sigma(1)}=\ddot{h}_{1}=\left\{0.2^{3 \times 0.15}, 0.4^{3 \times 0.15}, 0.5^{3 \times 0.15}\right\}=\{0.4847,0.6621,0.7320\} \\
\ddot{h}_{\sigma(2)}=\ddot{h}_{2}=\left\{0.2^{3 \times 0.3}, 0.6^{3 \times 0.3}\right\}=\{0.2349,0.6314\} \\
\ddot{h}_{\sigma(3)}=\ddot{h}_{3}=\left\{0.1^{3 \times 0.55}, 0.3^{3 \times 0.55}, 0.4^{3 \times 0.55}\right\}=\{0.0224,0.1372,0.2205\}
\end{gathered}
$$

$$
\operatorname{GHFHG}_{1}\left(h_{1}, h_{2}, h_{3}\right)=\operatorname{HFHG}\left(h_{1}, h_{2}, h_{3}\right)=\oplus_{i=1}^{3} \ddot{h}_{\sigma(i)}^{\omega_{i}}=\bigcup_{\ddot{\gamma}_{1} \in \ddot{h}_{1}, \dot{\gamma}_{2} \in \ddot{h}_{2}, \dot{\gamma}_{3} \in \dot{H}_{3}}\left\{\gamma_{1}^{0.3} \gamma_{2}^{0.4} \gamma_{3}^{0.3}\right\}
$$

$$
=\{0.1442,0.1584,0.1632,0.2142,0.2352,0.2424,0.2484,0.2728,0.2811
$$

$$
0.2864,0.3145,0.3241,0.3690,0.4051,0.4175,0.4254,0.4671,0.4814\}
$$

$\operatorname{GHFHG}_{3}\left(h_{1}, h_{2}, h_{3}\right)=\frac{1}{3}\left(\stackrel{3}{\otimes}\left(3 \ddot{h}_{\sigma(i)}\right)^{\omega_{i}}\right)$

$$
\begin{aligned}
= & \bigcup_{\ddot{\gamma}_{1} \in \ddot{h}_{1}, \dot{\gamma}_{2} \in \ddot{h}_{2}, \dot{\gamma}_{3} \in \ddot{H}_{3}}\left\{1-\left(1-\left(1-\left(1-\ddot{\gamma}_{1}\right)^{2}\right)^{0.25}\left(1-\left(1-\ddot{\gamma}_{2}\right)^{2}\right)^{0.4}\left(1-\left(1-\ddot{\gamma}_{3}\right)^{2}\right)^{0.35}\right)^{\frac{1}{3}}\right\} \\
= & \{0.1264,0.1312,0.1322,0.1633,0.1698,0.1710,0.2361,0.2467,0.2487, \\
& 0.2772,0.2905,0.2930,0.3222,0.3390,0.3423,0.3902,0.4138,0.4185\}
\end{aligned}
$$

In some practical problems, for example, the presidential election or the blind peer review of thesis, anonymity is required in order to protect the DMs' privacy or avoid influencing each other. In this section, we apply the hesitant fuzzy aggregation operators to MADM with anonymity. Suppose that there are $n$ alternatives $A_{i}(i=1,2, \cdots, n)$ and $m$ attributes $x_{j}(j=1,2, \cdots, m)$ with the attribute weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}}$ such that $w_{j} \in[0,1], j=1,2, \cdots, m$, and $\sum_{j=1}^{m} w_{j}=1$. If the DMs provide several values for the alternative $A_{i}$ under the attribute $x_{j}$ with anonymity, then these values can be considered as a HFE $h_{i j}$. In the case where two DMs provide the same value, then the value emerges only once in $h_{i j}$.

Based on the above analysis, we give the following decision making method (Xia and Xu 2011a):

Step 1. The DMs provide their evaluations about the alternative $A_{i}$ under the attribute $x_{j}$, denoted by the HFEs $h_{i j}(i=1,2, \cdots, n ; j=1,2, \cdots, m)$. In the process of aggregation, we may deal with two kinds of attributes, i.e., (1) The benefit type attributes, the bigger the preference values the better; (2) The cost type attributes, the smaller the preference values the better. In such cases, we may transform the preference values of the cost type attributes into the preference values of the benefit type attributes. Then $H=\left(h_{i j}\right)_{n \times m}$ can be transformed into the matrix $B=\left(b_{i j}\right)_{n \times m}$, where

$$
\begin{align*}
& b_{i j}=\bigcup_{t_{i j} \in b_{i j}}= \begin{cases}\left\{\gamma_{i j}\right\}, & \text { for benefit attribute } x_{j} \\
\left\{1-\gamma_{i j}\right\}, & \text { for cost attribute } x_{j}\end{cases} \\
& \qquad i=1,2, \ldots, n ; j=1,2, \ldots, m \tag{1.52}
\end{align*}
$$

and $\bigcup_{\gamma_{i j} h_{i j}}\left\{1-\gamma_{i j}\right\}=h_{i j}^{c}$ is the complement of $h_{i j}$.

Step 2. Utilize one of the developed aggregation operators to obtain the HFEs $b_{i}(i=1,2, \cdots, n)$ for the alternatives $A_{i}(i=1,2, \cdots, n)$, for example,

$$
\begin{equation*}
b_{i}=\operatorname{GHFWA}_{\lambda}\left(b_{i 1}, b_{i 2}, \cdots, b_{i m}\right)=\left(\bigoplus_{j=1}^{m} w_{j} b_{i j}^{\lambda}\right)^{\frac{1}{\lambda}} \tag{1.53}
\end{equation*}
$$

Step 3. Compute the scores $s\left(b_{i}\right)(i=1,2, \cdots, n)$ of $b_{i}(i=1,2, \cdots, n)$.

Step 4. Get the priority of the alternatives $A_{i}(i=1,2, \cdots, n)$ by ranking $s\left(b_{i}\right)(i=1,2, \cdots, n)$.

Example 1.9 (Parreiras et al. 2010). The enterprise's board of directors, which includes five members, is to plan the development of large projects (strategy initiatives) for the following five years. Suppose that there are four possible projects $A_{i}(i=1,2,3,4)$ to be evaluated. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance degrees, taking into account four attributes suggested by the Balanced Scorecard methodology (Kaplan and Norton 1996) (it should be noted that all of them are of benefit type): (1) $x_{1}$ : Financial perspective; (2) $x_{2}$ : The customer satisfaction; (3) $x_{3}$ : Internal business process perspective; (4) $x_{4}$ : Learning and growth perspective. Suppose that the weight vector of the attributes is $w=(0.2,0.3,0.15,0.35)^{\mathrm{T}}$.

In the following, we use the developed method to get the optimal project (Xia and Xu 2011a):

Step 1. In order to avoid influencing each other, the DMs are required to provide their preferences in anonymity and the decision matrix $H=\left(h_{i j}\right)_{4 \times 4}$ is presented in Table 1.1 (Xia and Xu 2011a), where $h_{i j}(i, j=1,2,3,4)$ are in the form of HFEs.

Table 1.1. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :--- |
| $A_{1}$ | $(0.2,0.4,0.7)$ | $(0.2,0.6,0.8)$ | $(0.2,0.3,0.6,0.7,0.9)$ | $(0.3,0.4,0.5,0.7,0.8)$ |
| $A_{2}$ | $(0.2,0.4,0.7,0.9)$ | $(0.1,0.2,0.4,0.5)$ | $(0.3,0.4,0.6,0.9)$ | $(0.5,0.6,0.8,0.9)$ |
| $A_{3}$ | $(0.3,0.5,0.6,0.7)$ | $(0.2,0.4,0.5,0.6)$ | $(0.3,0.5,0.7,0.8)$ | $(0.2,0.5,0.6,0.7)$ |
| $A_{4}$ | $(0.3,0.5,0.6)$ | $(0.2,0.4)$ | $(0.5,0.6,0.7)$ | $(0.8,0.9)$ |

Step 2. Considering that all the attributes $x_{j}(j=1,2,3,4)$ are the benefit type attributes, the preference values of the projects $A_{i}(i=1,2,3,4)$ do not need normalization. Thus, we utilize the GHFWA operator to obtain the HFEs $h_{i}(i=1,2,3,4)$ for the projects $A_{i}(i=1,2,3,4)$. Here we take the project $A_{4}$ for an example, and let $\lambda=1$, then
$h_{4}=\operatorname{GHFWA}_{1}\left(h_{41}, h_{42}, h_{43}, h_{44}\right)$
$=\operatorname{HFWA}(\{0.3,0.5,0.6\},\{0.2,0.4\},\{0.5,0.6,0.7\},\{0.8,0.9\})$
$=\oplus_{j=1}^{4} w_{j} h_{4 j}=\bigcup_{\gamma_{41} \in h_{41}, \gamma_{42} \in h_{42}, \gamma_{43} \in h_{43}, \gamma_{44} \in h_{44}}\left\{1-\prod_{j=1}^{4}\left(1-\gamma_{4 j}\right)^{w_{j}}\right\}$
$=\{0.5532,0.5679,0.5822,0.5861,0.5901,0.5960,0.6005,0.6036,0.6131$, $0.6136,0.6168,0.6203,0.6294,0.6299,0.6335,0.6450,0.6456,0.6494$, $0.6605,0.6610,0.6753,0.6722,0.6784,0.6830,0.6865,0.6890,0.6964$, $0.6969,0.6993,0.7021,0.7092,0.7097,0.7125,0.7215,0.7219,0.7337\}$

As the parameter $\lambda$ changes, we can get different results for each alternative. Here we will not list them for vast amounts of data.

Step 3. Compute the scores $s\left(h_{i}\right)(i=1,2,3,4)$ of $h_{i}(i=1,2,3,4)$. The scores for the alternatives $A_{i}(i=1,2,3,4)$ are shown in Table 1.2.

Step 4. By ranking $s\left(h_{i}\right)(i=1,2,3,4)$, we can get the priorities of the alternatives $A_{i}(i=1,2,3,4)$ as the parameter $\lambda$ changes, which are listed in Table 1.2 (Xia and Xu 2011a).

Table 1.2. Score values obtained by the GHFWA operator and the rankings of alternatives

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | Rankings |
| :--- | :---: | :---: | :---: | :---: | :---: |
| GHFWA $_{1}$ | 0.5634 | 0.6009 | 0.5178 | 0.6524 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| GHFWA $_{2}$ | 0.5847 | 0.6278 | 0.5337 | 0.6781 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| GHFWA $_{5}$ | 0.6324 | 0.6807 | 0.5723 | 0.7314 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| GHFWA $_{10}$ | 0.6730 | 0.7235 | 0.6087 | 0.7745 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| GHFWA $_{20}$ | 0.7058 | 0.7576 | 0.6410 | 0.8077 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |

From Table 1.2, we can find that the scores obtained by the GHFWA operator become bigger as the parameter $\lambda$ increases for the same aggregation arguments, and the DMs can choose the values of $\lambda$ according to their preferences.

In Step 2, if we use the GHFWG operator instead of the GHFWA operator to aggregate the values of the alternatives, the scores and the rankings of the alternatives are listed in Table 1.3 (Xia and Xu 2011a).

Table 1.3. Score values obtained by the GHFWG operator and the rankings of alternatives

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | Rankings |
| :--- | :---: | :---: | :---: | :---: | :---: |
| GHFWG $_{1}$ | 0.4783 | 0.4625 | 0.4661 | 0.5130 | $A_{4} \succ A_{1} \succ A_{3} \succ A_{2}$ |
| GHFWG $_{2}$ | 0.4546 | 0.4295 | 0.4526 | 0.4755 | $A_{4} \succ A_{1} \succ A_{3} \succ A_{2}$ |
| GHFWG $_{5}$ | 0.4011 | 0.3706 | 0.4170 | 0.4082 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| GHFWG $_{10}$ | 0.3564 | 0.3264 | 0.3809 | 0.3609 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| GHFWG $_{20}$ | 0.3221 | 0.2919 | 0.3507 | 0.3266 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |

It is pointed out that the ranking of the alternatives may change when the parameter $\lambda$ in the GHFWG operator changes. By analyzing Tables 1.2 and 1.3, we can find that the scores obtained by the GHFWG operator become smaller as the parameter $\lambda$ increases for the same aggregation arguments, but the values obtained by the GHFWA operator are always greater than the ones obtained by the GHFWG operator for the same value of the parameter $\lambda$ and the same aggregation values.

Although the HFHA (HFHG) operator generalizes both the HFWA (HFWG) and HFOWA (HFOWG) operators by weighting the given importance degrees and the ordered positions of the arguments, there is a flaw that the operator does not satisfy the desirable property, i.e., idempotency. An example can be used to illustrate this drawback.

Example 1.10 (Liao and Xu 2013 c ). Assume that $h_{1}=\{0.3,0.3,0.3\}, h_{2}=$ $\{0.3,0.3,0.3\}$ and $h_{3}=\{0.3,0.3,0.3\}$ are three HFEs, whose weight vector is $w=(1,0,0)^{\mathrm{T}}$, and the aggregation-associated vector is also $\omega=(1,0,0)^{\mathrm{T}}$, then

$$
\left.\begin{array}{rl}
\dot{h}_{1} & =3 \times 1 \times h_{1}=3 h_{1}=h_{1}=3 h_{1}=\left(1-(1-0.3)^{3}, 1-(1-0.3)^{3}, 1-(1-0.3)^{3}\right) \\
& =(0.657,0.657,0.657) \\
& \dot{h}_{2}
\end{array}=3 \times 0 \times h_{2}=0 \times h_{2}=\left(1-(1-0.3)^{0}, 1-(1-0.3)^{0}, 1-(1-0.3)^{0}\right)=(0,0,0)\right)
$$

Obviously, $s\left(\dot{h}_{1}\right)>s\left(\dot{h}_{2}\right)=s\left(\dot{h}_{3}\right)$. By using Eq.(1.48), we have

$$
\begin{aligned}
\operatorname{HFHA} & \left(h_{1}, h_{2}, h_{3}\right)=\oplus_{j=1}^{3}\left(\omega_{j} \dot{h}_{\sigma(j)}\right) \\
& =\bigcup_{\dot{\gamma}_{\sigma(1)} \in \dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)} \in \dot{h}_{\sigma(2)}, \dot{\gamma}_{\sigma(3)} \in \dot{h}_{\sigma(3)}}\left\{1-\left(1-\dot{\gamma}_{\sigma(1)}\right)^{1}\left(1-\dot{\gamma}_{\sigma(2)}\right)^{0}\left(1-\dot{\gamma}_{\sigma(3)}\right)^{0}\right\} \\
& =(0.657,0.657,0.657) \neq\{0.3,0.3,0.3\}
\end{aligned}
$$

Analogously,

$$
\begin{gathered}
\ddot{h}_{1}=h_{1}^{3 \times 1}=h_{1}^{3}=\left(0.3^{3}, 0.3^{3}, 0.3^{3}\right)=(0.027,0.027,0.027) \\
\ddot{h}_{2}=h_{2}^{3 \times 0}=h_{2}^{0}=\left(0.3^{0}, 0.3^{0}, 0.3^{0}\right)=(0,0,0) \\
\ddot{h}_{3}=h_{3}^{3 \times 0}=h_{3}^{0}=\left(0.3^{0}, 0.3^{0}, 0.3^{0}\right)=(0,0,0)
\end{gathered}
$$

$$
\begin{aligned}
\operatorname{HFHG}\left(h_{1}, h_{2}, h_{3}\right) & =\stackrel{3}{\otimes} \ddot{h}^{\stackrel{h}{h}^{(0)}} \omega_{\sigma(j)}=\bigcup_{\ddot{\gamma}_{\sigma(1)} \in \ddot{h}_{\sigma(1)}, \ddot{\gamma}_{\sigma(2)} \in \ddot{h}_{\sigma(2)}, \ddot{\gamma}_{\sigma(3)} \in \ddot{h}_{\sigma(3)}}\left\{\ddot{\gamma}_{\sigma(1)}^{1} \ddot{\gamma}_{\sigma(2)}^{0} \ddot{\gamma}_{\sigma(3)}^{0}\right\} \\
& =(0,0,0) \neq\{0.3,0.3,0.3\}
\end{aligned}
$$

Idempotency is one of the most important property for aggregation operators (Lin and Jiang 2011), but the HFHA and HFWG operators don't meet this basic property, we may develop some new hybrid aggregation operators which also weight the importance of each argument and its ordered position simultaneously. Recently, Liao and Xu (2013c) developed some new hybrid operators for HFEs:

Consider the HFOWA operator given as Eq.(1.45), it is equivalent to the following form:

$$
\begin{equation*}
\operatorname{HFOWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\oplus_{j=1}^{n}\left(\omega_{\sigma(j)} h_{j}\right) \tag{1.54}
\end{equation*}
$$

where $h_{j}$ is the $\sigma(j)$ th largest element of $h_{j}(j=1,2, \cdots, n)$. Inspired by this, suppose that the weighting vector of the elements is $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$, in order to weight the element and its position simultaneously, we can use such a form as $\oplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} h_{j}$, which weights both the element and its position. After normalization, a hesitant fuzzy hybrid arithmetical averaging operator can be generated as follows:

Definition 1.22 (Liao and Xu 2013c). For a collection of HFEs $h_{j}(j=1,2, \cdots, n)$, a hesitant fuzzy hybrid arithmetical averaging (HFHAA) operator is a mapping HFHAA: $\Theta^{n} \rightarrow \Theta$, defined by an associated weighting vector $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ with $\omega_{j} \in[0,1]$ and $\sum_{j=1}^{n} \omega_{j}=1$, such that

$$
\begin{equation*}
\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{\oplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}} \tag{1.55}
\end{equation*}
$$

where $\sigma:\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\}$ is the permutation such that $h_{j}$ is the $\sigma(j)$ th largest element of the collection of HFEs $h_{j}(j=1,2, \cdots, n)$, and
$w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weighting vector of the HFEs $h_{j}(j=1,2, \cdots, n)$, with $w_{j} \in[0,1], j=1,2, \cdots, n$, and $\sum_{j=1}^{n} w_{j}=1$.

Theorem 1.25 (Liao and Xu 2013c). For a collection of HFEs $h_{j}(j=1,2, \cdots, n)$, the aggregated value by using the HFHAA operator is also a HFE, and

$$
\begin{equation*}
\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right\} \tag{1.56}
\end{equation*}
$$

Proof. From the definition of HFE, it is obvious that the aggregated value by using the HFHAA operator is also a HFE.

By using the operational law (2) given in Definition 1.7, we have

$$
\begin{equation*}
\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}} h_{j}=\bigcup_{\gamma_{j} \in h_{j}}\left\{1-\left(1-\gamma_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right\}, j=1,2, \cdots, n \tag{1.57}
\end{equation*}
$$

Summing all these weighted HFEs $\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}} h_{j}(j=1,2, \cdots, n)$ by using the operational law (3) given in Definition 1.7, we can derive

$$
\begin{align*}
\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) & =\frac{\bigoplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}=\oplus_{j=1}^{n}\left(\bigcup_{\gamma_{j} \in h_{j}}\left\{1-\left(1-\gamma_{j}\right)^{\frac{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}{n} \omega_{\sigma(j)}}\right\}\right) \\
& =\bigcup_{\zeta_{1} \in h_{1}^{\prime}, \zeta_{2} \in h_{2}, \cdots, \zeta_{n} \in h_{n}^{\prime}}\left\{1-\prod_{j=1}^{n}\left(1-\zeta_{j}\right)\right\} \tag{1.58}
\end{align*}
$$

where

$$
\begin{equation*}
h_{j}^{\prime}=\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}} h_{j}, \zeta_{j}=1-\left(1-\gamma_{j} \sum^{\frac{\sum_{j=1}^{n} \omega_{j(j)}}{\sum_{j,(j)}}}, \gamma_{j} \in h_{j}, j=1,2, \cdots, n\right. \tag{1.59}
\end{equation*}
$$

Combining Eqs.(1.58) and (1.59) , we obtain

$$
\begin{align*}
\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) & =\underset{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}{\bigcup}\left\{1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\gamma_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right)\right)\right\} \\
& =\underbrace{}_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right\} \tag{1.60}
\end{align*}
$$

which completes the proof of Theorem 1.25.

Example 1.11 (Liao and Xu 2013c). Let $h_{1}=\{0.2,0.4,0.5\}, h_{2}=\{0.2,0.6\}$ and $h_{3}=\{0.1,0.3,0.4\}$ be three HFEs, whose weight vector is $w=(0.15,0.3,0.55)^{\mathrm{T}}$, and the aggregation-associated vector is $\omega=(0.3,0.4,0.3)^{\mathrm{T}}$.

At first, comparing $h_{1}, h_{2}$ and $h_{3}$ by using the score formula given as Definition 1.2, we have

$$
\begin{gathered}
s\left(h_{1}\right)=\frac{0.2+0.4+0.5}{3}=0.3667, s\left(h_{2}\right)=\frac{0.2+0.6}{2}=0.4 \\
s\left(h_{3}\right)=\frac{0.1+0.3+0.4}{3}=0.2667
\end{gathered}
$$

Since $s\left(h_{2}\right)>s\left(h_{1}\right)>s\left(h_{3}\right)$, then we can obtain $h_{2}>h_{1}>h_{3}$, and hence $\sigma(1)=2, \sigma(2)=1$, and $\sigma(3)=3$. Then

$$
\begin{gathered}
\frac{w_{1} \omega_{\sigma(1)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(j)}}=\frac{0.15 \times 0.4}{0.15 \times 0.4+0.3 \times 0.3+0.55 \times 0.3}=0.19 \\
\frac{w_{2} \omega_{\sigma(2)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(j)}}=0.286, \frac{w_{3} \omega_{\sigma(3)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(j)}}=0.524
\end{gathered}
$$

Then, by using Eq.(1.56), we can calculate that

$$
\begin{gathered}
\operatorname{HFHAA}\left(h_{1}, h_{2}, h_{3}\right)=\frac{\bigoplus_{j=1}^{3} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{1-\prod_{j=1}^{3}\left(1-\gamma_{j}\right)^{\substack{\begin{subarray}{c}{w_{j}, \sigma_{\sigma(j)} \\
j=1 j \sigma_{0}(j)} }}\end{subarray}}\right\} \\
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{1-\left(1-\gamma_{1}\right)^{0.19}\left(1-\gamma_{2}\right)^{0.286}\left(1-\gamma_{3}\right)^{0.524}\right\} \\
=\{0.1490,0.1943,0.2217,0.2541,0.2938,0.3020,0.3119,0.3178, \\
0.3392,0.3485,0.3617,0.3707,0.3882,0.4207,0.4356,0.4405, \\
0.4656,0.4838\}
\end{gathered}
$$

Theorem 1.26 (Liao and Xu 2013c). (Idempotency) If $h_{j}=h(j=1,2, \cdots, n)$, then $\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h$.

Proof. According to Eq.(1.55), we have

$$
\begin{equation*}
\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{\bigoplus_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}=\frac{\oplus_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)} h}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}=h \frac{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}=h \tag{1.61}
\end{equation*}
$$

Thus, HFHAA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h$, which completes the proof of the theorem.

Example 1.12 (Liao and Xu 2013c). Let's use the HFHAA operator to calculate Example 1.10. We have

$$
\begin{aligned}
\operatorname{HFHAA}\left(h_{1}, h_{2}, h_{3}\right) & =\frac{\stackrel{3}{j=1}_{\oplus}^{j=1} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} h_{2}, \gamma_{3} \in h_{3}}\left\{1-\left(1-\gamma_{1}\right)^{1}\left(1-\gamma_{2}\right)^{0}\left(1-\gamma_{3}\right)^{0}\right\} \\
& =\{0.3,0.3,0.3\}=h_{1}=h_{2}=h_{3}
\end{aligned}
$$

which satisfies the property of idempotency. This is also consistent with our intuition. From this example, we can see that the HFHAA operator is more reasonable than the HFHA operator developed by Xia and Xu (2011a).

By using the different manifestation of weighting vector, the HFHAA operator can reduce to some special cases. For example, if the associated weighting vector $\omega=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the HFHAA operator reduces to the HFWA operator; If $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the HFHAA operator reduces to the HFOWA operator. It must be pointed out that the weighting operation of the ordered position can be synchronized with the weighting operation of the given importance by the HFHAA operator. This characteristic is different from the HFHA operator.

Analogously, we also can develop the HFHAG operator for HFEs:
Definition 1.23 (Liao and Xu 2013c). For a collection of HFEs $h_{j}(j=1,2, \cdots, n)$, a hesitant fuzzy hybrid arithmetical geometric (HFHAG) operator is a mapping HFHAG: $\Theta^{n} \rightarrow \Theta$, defined by an associated weighting vector $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ with $\omega_{j} \in[0,1], j=1,2, \cdots, n$, and $\sum_{j=1}^{n} \omega_{j}=1$, such that

$$
\begin{equation*}
\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigotimes_{j=1}^{n}\left(h_{j}\right)^{\frac{\lambda_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}} \tag{1.62}
\end{equation*}
$$

where $\sigma:\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\}$ is the permutation such that $h_{j}$ is the $\sigma(j)$ th largest element of the collection of HFEs $h_{j}(j=1,2, \cdots, n)$, and
$w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weighting vector of the HFEs $h_{j}(j=1,2, \cdots, n)$, with $w_{j} \in[0,1], j=1,2, \cdots, n$, and $\sum_{j=1}^{n} w_{j}=1$.

Theorem 1.27 (Liao and Xu 2013c). For a collection of HFEs $h_{j}(j=1,2, \cdots, n)$, the aggregated value by using the HFHAG operator is also a HFE, and

$$
\begin{equation*}
\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{j=1}^{n} \gamma_{j}^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right\} \tag{1.63}
\end{equation*}
$$

Proof. Similar to Theorem 1.25, the aggregated value by using the HFHAG operator is also a HFE.

By using the operational law (1) given in Definition 1.7, we have

$$
\begin{equation*}
\left(h_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}=\bigcup_{\gamma \in h_{j}} \gamma^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}, j=1,2, \cdots, n \tag{1.64}
\end{equation*}
$$

According to the operational law (4) given in Definition 1.7, we can derive

$$
\begin{equation*}
\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\otimes_{j=1}^{n}\left(h_{j}\right)^{\frac{\sum_{j=1} \omega_{j} \omega_{\sigma(j)}}{\substack{n \\ w_{j} \omega_{\sigma(j)}}}}=\bigcup_{\xi_{1} \in h_{1}, \xi_{2} \in h_{2}^{n}, \cdots, \xi_{n} \in h_{n}^{\prime \prime}}\left\{\prod_{j=1}^{n} \bar{\xi}_{j}\right\} \tag{1.65}
\end{equation*}
$$

where

$$
\begin{equation*}
h_{j}^{\prime \prime}=\left(h_{j}\right)^{\substack{w_{j}, \omega_{\sigma(j)} \\ \sum_{j=1}, j \sigma_{\sigma(j)}}} \text { and } \bar{\xi}_{j}=\gamma_{j} \sum_{j=1}^{\frac{w_{j} \omega_{\sigma(j)}}{n} w_{j} \omega_{\sigma(j)}}, \gamma_{j} \in h_{j}, j=1,2, \cdots, n \tag{1.66}
\end{equation*}
$$

Combining Eqs.(1.65) and (1.66), we can obtain

$$
\begin{equation*}
\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{j=1}^{n} \gamma_{j}^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right\} \tag{1.67}
\end{equation*}
$$

which completes the proof of Theorem 1.27.
Example 1.13 (Liao and Xu 2013c). Let's use the HFHAG operator to fuse the HFEs $h_{1}, h_{2}$ and $h_{3}$ in Example 1.11. According to Theorem 1.27, we have

$$
\begin{aligned}
& \operatorname{HFHAG}\left(h_{1}, h_{2}, h_{3}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\prod_{j=1}^{3} \gamma_{j}^{\left.\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(j)}}\right\}}\right. \\
&= \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\gamma_{1}^{0.19} \gamma_{2}^{0.286} \gamma_{3}^{0.524}\right\} \\
&=\{0.1391,0.1587,0.1655,0.1904,0.2172,0.2266,0.2473,0.2822,0.2876,0.2944, \\
&0.3281,0.3387,0.3423,0.3863,0.3938,0.4031,0.4492,0.4686\}
\end{aligned}
$$

Theorem 1.28 (Idempotency) (Liao and Xu 2013c). If $h_{j}=h(j=1,2, \cdots, n)$, then $\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h$.

Proof. Since $h_{j}=h$, then $\gamma_{j}=\gamma$. Hence, according to Eq.(1.63), we have

$$
\begin{align*}
& \operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)= \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{j=1}^{n} \gamma_{j}^{\left.\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1} w_{j} \omega_{\sigma(j)}}\right\}}\right. \\
&=  \tag{1.68}\\
& \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}{\sum_{\gamma^{n=1}}^{n} w_{j} \omega_{\sigma(j)}}\right\}=\bigcup_{\gamma \in h}\{\gamma\}=h
\end{align*}
$$

Thus, HFHAG $\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h$, which completes the proof of Theorem 1.28.

Example 1.14 (Liao and Xu 2013c). Let's use the HFHAG operator to calculate Example 1.10, then we have

$$
\begin{aligned}
\operatorname{HFHAG}\left(h_{1}, h_{2}, h_{3}\right) & =\bigotimes_{j=1}^{n}\left(h_{j}\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\substack{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}} \\
= & \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\gamma_{1}^{1} \gamma_{2}^{0} \gamma_{3}^{0}\right\}=\{0.3,0.3,0.3\}=h_{1}=h_{2}=h_{3}
\end{aligned}
$$

which means the HFHAG operator satisfies idempotency, and thus is more reasonable than Xia and Xu (2011a)'s HFHG operator.

Especially, if the associated weighting vector $\omega=\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{\mathrm{T}}$, then the HFHAG operator reduces to the HFWG operator; If $w=\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{\mathrm{T}}$, then the HFHAG operator reduces to the HFOWG operator. With the HFHAG operator, the weighting operation of the ordered position also can be synchronized with the weighting operation of the given importance, while the HFHG operator does not have this characteristic.

Based on the HFHAA (HFHAG) operator, we can propose a procedure for the DM to select the best choice with hesitant fuzzy information, which involves the following steps (Liao and Xu 2013c):

Step 1. Construct the hesitant fuzzy decision matrix. The DM determines the relevant attributes of the potential alternatives and gives the evaluation information in the form of HFEs of the alternatives with respect to the attributes. When the DM is asked to compare the alternatives over attributes, he/she may have several possible values according to the sub-attributes. Thus, in this situation, it is natural to set out all the possible evaluations for an alternative under certain attributes given by the DM, which is represented as HFE. He/She also determines the importance degrees $w_{j}(j=1,2, \ldots, m)$ for the relevant attributes according to his/her preferences. Meanwhile, since different alternatives may have different focuses and advantages, to reflect this issue, the DM also gives the ordering weights $\omega_{j}(j=1,2, \ldots, m)$ for different attributes.

Step 2. Utilize the HFHAA operator (1.56) or the HFHAG operator (1.63) to obtain the HFEs $h_{i}(i=1,2, \cdots, n)$ for the alternatives $A_{i}(i=1,2, \cdots, n)$.

Step 3. Compute the scores $s\left(h_{i}\right)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ by Definition 1.2 and the deviation degrees $\bar{\sigma}^{\prime}\left(h_{i}\right)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ by Definition 1.5.

Step 4. Get the priority of the alternatives $A_{i}(i=1,2, \cdots, n)$ by ranking $s\left(h_{i}\right)$ and $\bar{\sigma}^{\prime}(h)(i=1,2, \cdots, n)$.

We now use a decision making problem to illustrate the procedure above:
Example 1.15 (Liao and Xu 2013c). Let's consider a customer who intends to buy a car. There are four alternatives $A_{i}(i=1,2,3,4)$ under consideration and the customer takes three attributes into account to determine which car to buy:
(1) $x_{1}$ : Quality of the car, which consists of three sub-attributes: $x_{11}$ : Safety, $x_{12}$ : Aerod. degree, and $x_{13}$ : Remedy for quality problems.
(2) $x_{2}$ : Overall cost of the product, which consists of four sub-attributes: $x_{21}$ : Product price, $x_{22}$ : Fuel economy, $x_{23}$ : Tax, and $x_{24}$ : Maintenance costs.
(3) $x_{3}$ : Appearance of the car, which consists of three sub-attributes: $x_{31}$ : Design; $x_{32}$ : Color, and $x_{33}$ : Comfort.

As mentioned above, it is appropriate for the customer to represent his/her preference assessments in HFEs to maintain the original evaluation information adequately, which are shown in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 3}$ (see Table 1.4 (Liao and Xu 2013c)). Note that the attributes have two different types such as benefit type and cost type. The customer should take this into account in the process of determining the preference values.

Table 1.4. Hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 3}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.6,0.7,0.9\}$ | $\{0.6,0.8\}$ | $\{0.3,0.6,0.9\}$ |
| $A_{2}$ | $\{0.7,0.9\}$ | $\{0.4,0.5,0.8,0.9\}$ | $\{0.4,0.8\}$ |
| $A_{3}$ | $\{0.6,0.8\}$ | $\{0.6,0.7,0.9\}$ | $\{0.3,0.5,0.7\}$ |
| $A_{4}$ | $\{0.6,0.8,0.9\}$ | $\{0.7,0.9\}$ | $\{0.2,0.4,0.7\}$ |

The weight information of these three attributes is also determined by the customer as $w=(0.5,0.3,0.2)^{\mathrm{T}}$. In addition, consider the fact that different cars may focus on different properties, for example, some cars are prominent in security with high price, while some cars are cheap but with low appearance. To reflect this concern, the customer gives another weight vector $\omega=(0.6,0.2,0.2)^{\mathrm{T}}$ for each attribute, which denotes that the most prominent feature of the car assigns more weight while the remainders assign less weight.

To select the most desirable car, here we utilize the HFHAA operator to obtain the HFEs $h_{i}$ for the cars $A_{i}(i=1,2,3,4)$. Now we take $A_{2}$ as an example, then
$h_{2}=\operatorname{HFHAA}\left(h_{21}, h_{22}, h_{23}\right)=\operatorname{HFHAA}(\{0.7,0.9\},\{0.4,0.5,0.8,0.9\},\{0.4,0.8\})$

Since

$$
\begin{gathered}
s\left(h_{21}\right)=\frac{0.7+0.9}{2}=0.8, s\left(h_{22}\right)=\frac{0.4+0.5+0.8+0.9}{4}=0.65 \\
s\left(h_{23}\right)=\frac{0.4+0.8}{2}=0.6
\end{gathered}
$$

then $h_{21}>h_{22}>h_{23}$. Thus, $\sigma(21)=1, \sigma(22)=2, \sigma(23)=3$. Then,

$$
\begin{gathered}
\frac{w_{1} \omega_{\sigma(21)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=\frac{0.5 \times 0.6}{0.5 \times 0.6+0.3 \times 0.2+0.2 \times 0.2}=0.75 \\
\frac{w_{2} \omega_{\sigma(22)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=0.15, \frac{w_{3} \omega_{\sigma(23)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=0.1
\end{gathered}
$$

Thus, by using Eq.(1.56), we can calculate that

$$
h_{2}=\operatorname{HFHAA}\left(h_{21}, h_{22}, h_{23}\right)=\operatorname{HFHAA}(\{0.7,0.9\},\{0.4,0.5,0.8,0.9\},\{0.4,0.8\})
$$

$$
\begin{aligned}
& =\bigcup_{\gamma_{21} \in h_{21}, \gamma_{22} \in h_{22}, \gamma_{23} \in h_{23}}\left\{1-\left(1-\gamma_{21}\right)^{0.75}\left(1-\gamma_{22}\right)^{0.15}\left(1-\gamma_{23}\right)^{0.1}\right\} \\
& =\{0.6423,0.6529,0.6848,0.6890,0.6974,0.7273,0.7289,0.7557,0.8435 \\
& 0.8477,0.8598,0.8636,0.8673,0.8804,0.8811,0.8928\}
\end{aligned}
$$

Similarly, we can calculate different results by using the HFHAA operator for other alternatives, $A_{1}, A_{3}$, and $A_{4}$. Here we will not list them for vast amounts of data.

Finally, we can compute the scores $s\left(h_{i}\right)(i=1,2,3,4)$ and the deviation degrees $\bar{\sigma}^{\prime}\left(h_{i}\right)(i=1,2,3,4)$ of $h_{i}(i=1,2,3,4)$. By ranking $s\left(h_{i}\right)(i=1,2,3,4)$, we can get the priorities of the alternatives $A_{i}(i=1,2,3,4)$. Since $s\left(h_{1}\right)=0.7329, s\left(h_{2}\right)=0.6953, s\left(h_{3}\right)=0.716$, and $s\left(h_{4}\right)=0.7782$, we get $s\left(h_{4}\right)>s\left(h_{1}\right)>s\left(h_{3}\right)>s\left(h_{2}\right)$, then $h_{4} \succ h_{1} \succ h_{3} \succ h_{2}$, i.e., the car $A_{4}$ is the most desirable choice for the customer.

If we use Xia and Xu (2011a)'s HFHA operator to solve this problem, then we have
$h_{1}=\operatorname{HFHA}\left(h_{11}, h_{12}, h_{13}\right)=\operatorname{HFHA}(\{0.6,0.7,0.9\},\{0.6,0.8\},\{0.3,0.6,0.9\})$
$=\{0.6438,0.6670,0.7180,0.6856,0.7060,0.7511,0.7251,0.7429,0.7823$, $0.7573,0.7731,0.8079,0.8977,0.9044,0.9190,0.9097,0.9156,0.9285\}$

$$
\begin{aligned}
h_{2}= & \operatorname{HFHA}\left(h_{21}, h_{22}, h_{23}\right)=\operatorname{HFHA}(\{0.7,0.9\},\{0.4,0.5,0.8,0.9\},\{0.4,0.8\}) \\
= & \{0.7097,0.7456,0.7191,0.7538,0.7618,0.7912,0.7897,0.8157,0.8920, \\
& 0.9053,0.8955,0.9084,0.9114,0.9223,0.9218,0.9314\}
\end{aligned}
$$

$h_{3}=\operatorname{HFHA}\left(h_{31}, h_{32}, h_{33}\right)=\operatorname{HFHA}(\{0.6,0.8\},\{0.6,0.7,0.9\},\{0.3,0.5,0.7\})$
$=\{0.6438,0.6579,0.6783,0.6618,0.6752,0.6945,0.7225,0.7335,0.7493$,
$0.8091,0.8167,0.8276,0.8188,0.8259,0.8363,0.8513,0.8572,0.8657\}$

$$
\begin{aligned}
h_{4}= & \operatorname{HFHA}\left(h_{41}, h_{42}, h_{43}\right)=\operatorname{HFHA}(\{0.6,0.8,0.9\},\{0.7,0.9\},\{0.2,0.4,0.7\}) \\
= & \{0.6564,0.6680,0.6945,0.7180,0.7276,0.7493,0.8158,0.8221,0.8363, \\
& 0.8489,0.8540,0.8657,0.9013,0.9047,0.9123,0.9190,0.9218,0.9280\}
\end{aligned}
$$

Since $s\left(h_{1}\right)=0.7908, s\left(h_{2}\right)=0.8359, s\left(h_{3}\right)=0.7625$, and $s\left(h_{4}\right)=0.8191$, we get $s\left(h_{2}\right)>s\left(h_{4}\right)>s\left(h_{1}\right)>s\left(h_{3}\right)$, then $h_{2}>h_{4}>h_{1}>h_{3}$. With Xia and

Xu (2011a)'s HFHA operator, the car $A_{2}$ turns out to be the most desirable choice for the customer, and all the other cars are in the same rank. Meanwhile, when using Xia and Xu (2011a)'s HFHA operator, we first need to calculate $\dot{h}=n w_{k} h_{k}$ and compare them, and then calculate $\omega_{j} \dot{h}_{\sigma(j)}$, after which, we shall compute the aggregation values with $\oplus_{j=1}^{n}\left(\omega_{j} \dot{h}_{\sigma(j)}\right)$. Since the computation with HFEs is very complex, the results derived via Xia and Xu (2011a)'s HFHA operator is hard to be obtained. As for the HFHAA operator, the weighting operation of the ordered position is synchronized with the weighting operation of the given importance, which is in the mathematical form as $w_{j} \omega_{\sigma(j)}$. Since both
$w_{j}$ and $\omega_{\sigma(j)}$ are crisp numbers, we only need to calculate $\frac{\bigoplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}$,
which makes the HFHAA operator easier to calculate than Xia and Xu (2011a)'s HFHA operator.

### 1.3 Hesitant Fuzzy Bonferroni Means

Bonferroni firstly introduced the BM (Bonferroni 1950), which can provide for the aggregation lying between the max and min operators and the logical "or" and "and" operators. Since the BM can capture the expressed interrelationship of the individual arguments, it has been receiving much attention from researchers over the last decades. Yager (2009) gave an interpretation of this operator and suggested some generalizations which replace the simple average by other mean type operators. Beliakov et al. (2010) gave a systematic investigation of a family of composed aggregation functions which also generalize the BM. To overcome the limitation that the BM can only take the forms of crisp numbers rather than any other types of arguments, Xu and Yager (2011) investigated the BM under intuitionistic fuzzy environment, Xia et al. (2013b) proposed the generalized intuitionistic fuzzy BMs and the geometric BMs.

Definition 1.24 (Bonferroni 1950). Let $p, q \geq 0$, and $a_{i}(i=1,2, \ldots, n)$ be a collection of nonnegative numbers, if

$$
\begin{equation*}
B^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\left(\frac{1}{n(n-1)} \sum_{\substack{i, j=1 \\ i \neq j}}^{n} a_{i}^{p} a_{j}^{q}\right)^{\frac{1}{p+q}} \tag{1.69}
\end{equation*}
$$

then $B^{p, q}$ is called a Bonferroni mean (BM).

Apparently, the BM has the following properties:
(1) $B^{p, q}(0,0, \ldots, 0)=0$.
(2) $B^{p, q}(a, a, \ldots, a)=a$, if $a_{i}=a$, for all $i$.
(3) $B^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \geq B^{p, q}\left(d_{1}, d_{2}, \ldots, d_{n}\right)$, i.e., $B^{p, q}$ is monotonic, if $a_{i} \geq d_{i}$, for all $i$.
(4) $\min \left\{a_{i}\right\} \leq B^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq \max \left\{a_{i}\right\}$.

Zhu et al. (2013a) combined the BM with the hesitant fuzzy information represented by HFEs and developed the hesitant fuzzy Bonferroni mean (HFBM):

Definition 1.25 (Zhu et al. 2013a). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs. For any $p, q>0$, if

$$
\begin{equation*}
H F B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\left(\frac { 1 } { n ( n - 1 ) } \left(\underset{\substack{\begin{subarray}{c}{i, j=1 \\
i \neq j} }}\end{subarray}}{\left.\left.\stackrel{n}{\oplus}\left(h_{i}^{p} \otimes h_{j}^{q}\right)\right)\right)^{\frac{1}{p+q}}{ }^{p}}\right.\right. \tag{1.70}
\end{equation*}
$$

then we call $H F B^{p, q}$ a hesitant fuzzy Bonferroni mean (HFBM).
We can further get the following theorem:
Theorem 1.29 (Zhu et al. 2013a). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, then the aggregated value by using the HFBM is a HFE, and

$$
\begin{equation*}
\operatorname{HFB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}, i \neq j}\left\{\left(1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\} \tag{1.71}
\end{equation*}
$$

where $\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}$ reflects the interrelationship between $h_{i}$ and $h_{j}, i, j=$ $1,2, \ldots, n, i \neq j$.

Proof. Since

$$
\begin{equation*}
\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}}\left\{\gamma_{i, j}\right\}=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{\gamma_{i}^{p} \gamma_{j}^{q}\right\} \tag{1.72}
\end{equation*}
$$

which is also a HFE, and

$$
\begin{equation*}
\operatorname{HFB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\ i \neq j}}^{n} \sigma_{i, j}\right)\right)^{\frac{1}{p+q}} \tag{1.73}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\ i \neq j}}^{n} \sigma_{i, j}\right)=\bigoplus_{\substack{i, j=1 \\ i \neq j}}^{n}\left(\frac{1}{n(n-1)} \sigma_{i, j}\right)=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}}\left\{1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}\right)^{\frac{1}{n(n-1)}}\right\} \tag{1.74}
\end{equation*}
$$

then

$$
\begin{equation*}
\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\ i \neq j}}^{n} \sigma_{i, j}\right)\right)^{\frac{1}{p+q}}=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}, i \neq j}\left(1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.75}
\end{equation*}
$$

where $\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{\gamma_{i}^{p} \gamma_{j}^{q}\right\}$, which completes the proof.
Below we further discuss some desirable properties of the HFBM, i.e., the monotonicity, the commutativity and the boundedness (Zhu et al. 2013a):
(1) (Monotonicity). Let $\left\{h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right\}$ and $\left\{h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right\}$ be two collections of HFEs, if for any $\gamma_{\alpha_{i}} \in h_{\alpha_{i}}$ and $\gamma_{\beta_{i}} \in h_{\beta_{i}}$, we have $\gamma_{\alpha_{i}} \leq \gamma_{\beta_{i}}$ for all $i$, and $\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}}\left\{\gamma_{i, j}\right\}=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{\gamma_{i}^{p} \gamma_{j}^{q}\right\}$, then

$$
\begin{equation*}
H F B^{p, q}\left(h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right) \leq \operatorname{HFB}^{p, q}\left(h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right) \tag{1.76}
\end{equation*}
$$

Proof. Since $\gamma_{\alpha_{i}} \leq \gamma_{\beta_{i}}, \quad$ for any $\gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \quad \gamma_{\beta_{i}} \in h_{\beta_{i}}, i \neq j$, then $\gamma_{\alpha_{i}} \gamma_{\alpha_{j}} \leq \gamma_{\beta_{i}} \gamma_{\beta_{j}}$ and for any $\gamma_{\alpha_{i, j}} \in \sigma_{\alpha_{i, j}}, \quad \gamma_{\beta_{i, j}} \in \sigma_{\beta_{i, j}}, i \neq j$, we have $\gamma_{\alpha_{i, j}}=\gamma_{\alpha_{i}}^{p} \gamma_{\alpha_{j}}^{A} \leq \gamma_{\beta_{i, j}}=\gamma_{\beta_{i}}^{p} \gamma_{\beta_{j}}^{A}$. Thus,

$$
\begin{equation*}
\left(1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{\alpha_{i, j}}\right) \frac{1}{n(n-1)}\right)^{\frac{1}{p+q}} \leq\left(1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{\beta_{i, j}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.77}
\end{equation*}
$$

we have

$$
\begin{align*}
& H F B^{p, q}\left(h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right)=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\
i \neq j}}^{\substack{\alpha_{\alpha_{i, j}}}}\right)\right)^{\frac{1}{p+q}} \\
& \quad=\bigcup_{\gamma_{\alpha_{i, j}} \in \sigma_{\alpha_{i, j}, i \neq j}}\left(1-\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\gamma_{\alpha_{i, j}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \\
& \quad \leq \bigcup_{\gamma_{\beta_{i, j}} \in \sigma_{\beta_{i, j}, i \neq j}}\left(1-\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\gamma_{\beta_{i, j}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}=\left(\frac { 1 } { n ( n - 1 ) } \left({\left.\left.\underset{\substack{i, j=1 \\
i \neq j}}{n} \sigma_{\beta_{i, j}}\right)\right)^{\frac{1}{p+q}}}_{\quad=H F B^{p, q}\left(h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right)}\right.\right.
\end{align*}
$$

which completes the proof.
(2) (Commutativity). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, and $\left(\dot{h}_{1}, \dot{h}_{2}, \ldots, \dot{h}_{n}\right)$ any permutation of $\left(h_{1}, h_{2}, \ldots, h_{n}\right)$, then

$$
\begin{align*}
\operatorname{HFB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right) & =\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\
i \neq j}}^{n} \sigma_{i, j}\right)\right)^{\frac{1}{p+q}} \\
& =\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\
i \neq j}}^{n} \dot{\sigma}_{i, j}\right)\right)^{\frac{1}{p+q}}=\operatorname{HFB}^{p, q}\left(\dot{h}_{1}, \dot{h}_{2}, \ldots, \dot{h}_{n}\right) \tag{1.79}
\end{align*}
$$

where $\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}$ and $\dot{\sigma}_{i, j}=\dot{h}_{i}^{p} \otimes \dot{h}_{j}^{q}(i, j=1,2, \ldots, n, i \neq j)$.
(3) (Boundedness). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs, $h_{i}^{+}=$

$$
\begin{array}{r}
\bigcup \begin{array}{l}
\gamma_{i} \in h_{i} \\
\max \left\{\gamma_{i}\right\}, h_{i}^{-}=\bigcup_{\gamma_{i} \in h_{i}} \min \left\{\gamma_{i}\right\}, \gamma_{i}^{+} \in h_{i}^{+}, \gamma_{i}^{-} \in h_{i}^{-}, \text {and } \\
\sigma_{i, j}
\end{array}=h_{i}^{p} \otimes h_{j}^{q}=\bigcup_{\gamma_{i, j} \in \sigma_{i, j}}^{\bigcup}\left\{\gamma_{i, j}\right\}=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{\gamma_{i}^{p} \gamma_{j}^{q}\right\}
\end{array}
$$

then

$$
\begin{equation*}
h_{i}^{-} \leq H F B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h_{i}^{+} \tag{1.81}
\end{equation*}
$$

Proof. Since $\gamma_{i}^{-} \leq \gamma_{i} \leq \gamma_{i}^{+}$, for all $i$, then

$$
\begin{gather*}
\left(\gamma_{i}^{-}\right)^{p+q} \leq \gamma_{i}^{p} \gamma_{j}^{q} \leq\left(\gamma_{i}^{+}\right)^{p+q}  \tag{1.82}\\
\left(\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\gamma_{i, j}\right)\right)^{\frac{1}{n(n-1)}} \geq\left(\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\left(\gamma_{i}^{+}\right)^{p+q}\right)\right)^{\frac{1}{n(n-1)}}  \tag{1.83}\\
\left(\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\gamma_{i, j}\right)\right)^{\frac{1}{n(n-1)}} \leq\left(\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\left(\gamma_{i}^{-}\right)^{p+q}\right)\right)^{\frac{1}{n(n-1)}} \tag{1.84}
\end{gather*}
$$

and thus

$$
\begin{equation*}
\left(1-\left(\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \geq\left(1-\left(1-\left(\gamma_{i}^{-}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}=\gamma_{i}^{-} \tag{1.85}
\end{equation*}
$$

$$
\begin{equation*}
\left(1-\left(\prod_{\substack{i,=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq\left(1-\left(1-\left(\gamma_{i}^{+}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}=\gamma_{i}^{+} \tag{1.86}
\end{equation*}
$$

we have
which completes the proof.
The advantage of the HFBM is that it can capture the interrelationship between HFEs, which can be characterized by a function denoted by

$$
\begin{equation*}
\tau_{i, j}=\sigma_{i, j} \oplus \sigma_{j, i}=\left(h_{i}^{p} \otimes h_{j}^{q}\right) \oplus\left(h_{j}^{p} \otimes h_{i}^{q}\right)(i, j=1,2, \ldots, n, i<j) \tag{1.89}
\end{equation*}
$$

where $\sigma_{i, j}=h_{i}^{p} \otimes h_{j}^{q}$ and $\sigma_{j, i}=h_{j}^{p} \otimes h_{i}^{q}$. Apparently, $\tau_{i, j}$ represents the interrelationship between the HFEs $h_{i}$ and $h_{j}$, and $\tau_{i, j}$ is also a HFE. We consider $\tau_{i, j}$ as a "bonding satisfaction" factor used as a calculation unit in the HFBM, and Zhu et al. (2013a) called it a hesitant Bonferroni element (HBE).

On the basis of the HBE, we have
and can further develop the following theorem:

Theorem 1.30 (Zhu et al. 2013a). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, and $\tau_{i, j}(i, j=1,2, \ldots, n, i<j)$ the HBE, then

$$
\begin{equation*}
\operatorname{HFB}^{p, q}\left(h_{1}, h_{2,}, \ldots, h_{n}\right)=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j}, i<j}\left\{\left(1-\prod_{\substack{i, j=1 \\ i<j}}^{n}\left(1-\varepsilon_{i, j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\} \tag{1.91}
\end{equation*}
$$

where $\tau_{i, j}=\left(h_{i}^{p} \otimes h_{j}^{q}\right) \oplus\left(h_{j}^{p} \otimes h_{i}^{q}\right)(i, j=1,2, \ldots, n, i<j)$.
Moreover, if we exchange the parameters $p$ and $q$, we can get a new property called "idempotent commutativity", which is stated as follows:
(4) (Idempotent Commutativity). Since

$$
\begin{equation*}
\tau_{i, j}=\left(h_{i}^{p} \otimes h_{j}^{q}\right) \oplus\left(h_{j}^{p} \otimes h_{i}^{q}\right)=\left(h_{i}^{q} \otimes h_{j}^{p}\right) \oplus\left(h_{j}^{q} \otimes h_{i}^{p}\right)(i, j=1,2, \ldots, n, i<j) \tag{1.92}
\end{equation*}
$$

then by exchanging the parameters $p$ and $q$, we have

$$
\begin{align*}
\operatorname{HFB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right) & =\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\
i<j}}^{n} \tau_{i, j}\right)\right)^{\frac{1}{p+q}} \\
& =\left(\frac { 1 } { n ( n - 1 ) } \left({\left.\left.\underset{\substack{i, j=1 \\
i<j}}{n} \tau_{i, j}\right)\right)^{\frac{1}{q+p}}=\operatorname{HFB}^{q, p}\left(h_{1}, h_{2,}, h_{n}\right)}^{n}=\right.\right. \tag{1.93}
\end{align*}
$$

where $\tau_{i, j}=\left(h_{i}^{p} \otimes h_{j}^{q}\right) \oplus\left(h_{j}^{q} \otimes h_{i}^{p}\right)(i, j=1,2, \ldots, n, i<j)$.

Now, we give an example to illustrate our results:
Example 1.16 (Zhu et al. 2013a). Assume that we have three HFEs, $h_{1}=\{0.1\}$, $h_{2}=\{0.2,0.4\}$ and $h_{3}=\{0.3\}$. Then based on the operations of HFEs, we have

$$
\begin{aligned}
& h_{1} \otimes h_{2}=\{0.02,0.04\}, h_{2} \otimes h_{1}=\{0.02,0.04\}, h_{1} \otimes h_{3}=\{0.03\} \\
& h_{3} \otimes h_{1}=\{0.03\}, h_{2} \otimes h_{3}=\{0.06,0.12\}, h_{3} \otimes h_{2}=\{0.06,0.12\}
\end{aligned}
$$

and obtain

$$
\begin{aligned}
H F B^{1,1}\left(h_{1}, h_{2}, h_{3}\right)= & \{0.1919,0.2003,0.2176,0.2250,0.2084,0.2321, \\
& 0.2403,0.2470,0.2534\}
\end{aligned}
$$

$H F B^{2,2}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2011,0.2069,0.2423,0.2457,0.2124,0.2490$, $0.2692,0.2718,0.2742\}$
$H F B^{1,6}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2360,0.2712,0.2523,0.2792,0.2361,0.2713$, $0.3045,0.3086,0.3153,0.3187\}$
$H F B^{6,1}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2360,0.2712,0.2523,0.2792,0.2361,0.2713$, $0.3045,0.3086,0.3153,0.3187\}$
$H F B^{1,0}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2042,0.2414,0.2770\}$
$H F B^{0,1}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2042,0.2414,0.2770\}$

Then we get

$$
\begin{aligned}
& s\left(H F B^{1,1}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2242, s\left(\operatorname{HFB}^{1,6}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2857 \\
& s\left(H F B^{1,0}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2401, s\left(\operatorname{HFB}^{2,2}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2425 \\
& s\left(H F B^{1,6}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2857, s\left(H F B^{0,1}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.2401
\end{aligned}
$$

When the values of the parameters $p$ and $q$ change, more details can be found in Figs. 1.1-1.4 (Zhu et al. 2013a).


Fig. 1.1. Scores of the HFBM $(p=q, p \in[1,10])$


|  |  |  |  | 1 | 1 | 1 | 1 |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0.28 | 0.285 | 0.29 | 0.295 | 0.3 | 0.305 | 0.31 | 0.315 | 0.32 | 0.325 |  |

Fig. 1.2. Scores of the HFBM $(p=q+5, q \in[1,10])$


Fig. 1.3. Scores of the HFBM $(q=p+5, p \in[1,10])$


Fig. 1.4. Scores of the $\operatorname{HFBM}(q=0, p \in[1,10] ; p=0, q \in[1,10])$

Next, we discuss some special cases by changing the parameters $p$ and $q$ as follows (Zhu et al. 2013a):

Case 1. If $q \rightarrow 0$, then we obtain a generalized hesitant fuzzy mean (GHFM):

$$
\begin{equation*}
\operatorname{HFB}^{p, 0}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\left(\frac{1}{n}\left(\oplus_{i=1}^{n} h_{i}^{p}\right)\right)^{\frac{1}{p}}=\bigcup_{\gamma_{i} \in h_{i}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{p}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right\} \tag{1.94}
\end{equation*}
$$

Case 2. If $p=2$ and $q \rightarrow 0$, then we obtain a hesitant fuzzy square mean (HFSM):

$$
\begin{equation*}
H F B^{2,0}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\left(\frac{1}{n}\left(\oplus_{i=1}^{n} h_{i}^{2}\right)\right)^{\frac{1}{2}}=\bigcup_{\gamma_{i} \in h_{i}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{i}^{2}\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right\} \tag{1.95}
\end{equation*}
$$

Case 3. If $p=1$ and $q \rightarrow 0$, then we obtain the HFA operator (Xia and Xu 2011a):

Case 4. If $p=q=1$, then we obtain a hesitant fuzzy interrelated square mean (HFISM):

$$
\begin{align*}
H F B^{1,1}\left(h_{1}, h_{2}, \ldots, h_{n}\right) & =\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i, j=1 \\
i \neq j}}^{n}\left(h_{i} \otimes h_{j}\right)\right)\right)^{\frac{1}{2}} \\
& =\bigcup_{\delta_{i, j} \in \rho_{i, j}, i \neq j}\left\{\left(1-\prod_{\substack{i, j=1 \\
i \neq j}}^{n}\left(1-\delta_{i, j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right\} \tag{1.97}
\end{align*}
$$

where $\rho_{i, j}=h_{i} \otimes h_{j}(i, j=1,2, \ldots, n, i \neq j)$.

To consider the importance of aggregated arguments, we develop the weighted hesitant fuzzy Bonferroni mean (WHFBM) as follows:

Definition 1.26 (Zhu et al. 2013a). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$ their weight vector, where $w_{i}$ indicates the importance degree of $h_{i}$, satisfying $w_{i} \in[0,1], i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$, if

$$
\begin{equation*}
W H F B_{w}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\left(\frac{1}{n(n-1)}\left(\underset{\substack{i, j=1 \\ i \neq j}}{n}\left(\left(w_{i} h_{i}\right)^{p} \otimes\left(w_{j} h_{j}\right)^{q}\right)\right)\right)^{\frac{1}{p+q}} \tag{1.98}
\end{equation*}
$$

then we call $H F B_{w}^{p, q}$ a weighted hesitant fuzzy Bonferroni mean (WHFBM).

Theorem 1.31 (Zhu et al. 2013a). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, whose weight vector is $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$, which satisfies $w_{i} \in[0,1], i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$, then the aggregated value by using the WHFBM is a HFE, and

$$
\begin{equation*}
W_{H F B}{ }^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{i, j}^{\prime \prime} \sigma_{i, j}^{w}, i \neq j}\left\{\left(1-\prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(1-\gamma_{i, j}^{w}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\} \tag{1.99}
\end{equation*}
$$

where $\sigma_{i, j}^{w}=\left(w_{i} h_{i}\right)^{p} \otimes\left(w_{j} h_{j}\right)^{q}$ reflects the interrelationship between $h_{i}$ and $h_{j}(i, j=1,2, \ldots, n, i \neq j)$.

As the typical applications of the aggregation operators, we now develop an approach for MADM based on the WHFBM (Zhu et al. 2013a):

Step 1. For a MADM problem, let $A=\left\{A_{1}, A_{2}, \ldots, A_{n}\right\}$ be a set of $n$ alternatives, $X=\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$ a set of $m$ attributes, whose weight vector is $w=\left(w_{1}, w_{2}, \ldots, w_{m}\right)^{\mathrm{T}}$, satisfying $w_{j}>0, j=1,2, \ldots, m$, and $\sum_{j=1}^{m} w_{j}=1$, where $w_{j}$ denotes the importance degree of the attribute $x_{j}$. The DMs provide
all the possible values that the alternative $A_{i}$ satisfies the attribute $x_{j}$ represented by the HFEs $h_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\gamma_{i j}\right\}(i=1,2, \ldots, n ; j=1,2, \ldots, m)$, which are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{n \times m}$ (see Table 1.5 (Zhu et al. 2013a)).

Table 1.5. The hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $\cdots$ | $x_{m}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $h_{11}$ | $h_{12}$ | $\cdots$ | $h_{1 m}$ |
| $A_{2}$ | $h_{21}$ | $h_{22}$ | $\cdots$ | $h_{2 m}$ |
| $\ldots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $A_{n}$ | $h_{n 1}$ | $h_{n 2}$ | $\cdots$ | $h_{n m}$ |

Step 2. To normalize the preference information, we first transform $H=\left(h_{i j}\right)_{n \times m}$ into the matrix $B=\left(b_{i j}\right)_{n \times m}$ by using the formula (1.52), and then utilize the WHFBM (in general, we can take $p \neq 0$, and $q \neq 0$ ) to aggregate all the preference values $b_{i j}(j=1,2, \ldots, n)$ of the $i$ th line and get the overall performance value $b_{i}$ corresponding to the alternative $A_{i}$ :

$$
\begin{equation*}
b_{i}=W H F B_{w}^{p, q}\left(b_{i 1}, b_{i 2}, \ldots, b_{i m}\right) \tag{1.100}
\end{equation*}
$$

Step 3. Calculate the scores $s\left(b_{i}\right)(i=1,2, \ldots, n)$ of $b_{i}(i=1,2, \ldots, n)$ and rank all the alternatives $A_{i}(i=1,2, \ldots, n)$ according to $s\left(b_{i}\right)(i=1,2, \ldots, n)$ in descending order.

In the following, we apply our approach to a MADM problem.
Example 1.17 (Zhu et al. 2013a). Let us consider a site selection problem. Three alternatives $A_{i}(i=1,2,3)$ are available. The DMs consider three attributes: $x_{1}$ (price), $x_{2}$ (location), $x_{3}$ (environment). Let the weight vector of the attributes $x_{j}(j=1,2,3)$ be $w=(0.5,0.3,0.2)^{\mathrm{T}}$. Assume that the characteristics of the alternatives $A_{i}(i=1,2,3)$ with respect to the attributes $x_{j}(j=1,2,3)$ are represented by the HFEs $h_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\gamma_{i j}\right\}$, where $\gamma_{i j}$ indicates the degree that the
alternative $A_{i}$ satisfies the attribute $x_{j}$. All $h_{i j}(i=1,2,3 ; j=1,2,3)$ are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{3 \times 3}$ (see Table 1.6 (Zhu et al. 2013a)).

Table 1.6. The hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.6,0.7,0.8\}$ | $\{0.25\}$ | $\{0.4,0.5\}$ |
| $A_{2}$ | $\{0.4\}$ | $\{0.4,0.5\}$ | $\{0.3,0.55,0.6\}$ |
| $A_{3}$ | $\{0.2,0.4\}$ | $\{0.6,0.5\}$ | $\{0.7,0.5\}$ |

Considering that all the attributes $x_{j}(j=1,2,3)$ are the benefit type attributes, the preference values of the alternatives $A_{i}(i=1,2,3)$ do not need normalization. We utilize the WHFBM (here, we take $p=q=1$ ) to aggregate all the preference values $h_{i j}(j=1,2,3)$ of the $i$ th line and get the overall performance value $h_{i}$ with respect to the alternative $A_{i}$ as:

$$
\begin{gathered}
h_{1}=\{0.1483,0.1514, \ldots, 0.1976\}, h_{2}=\{0.1383,0.1446, \ldots, 0.1939\} \\
h_{3}=\{0.1818,0.1953, \ldots, 0.1788\}
\end{gathered}
$$

Then, we calculate the scores of all the alternatives as:

$$
s\left(h_{1}\right)=0.1820, s\left(h_{2}\right)=0.1703, s\left(h_{3}\right)=0.1828
$$

Since $s\left(h_{3}\right)>s\left(h_{1}\right)>s\left(h_{2}\right)$, then we get the ranking of the HFEs as $h_{3}>h_{1}>h_{2}$, consequently the ranking of the alternatives $A_{i}(i=1,2,3)$ is $A_{3} \succ A_{1} \succ A_{2}$. Thus, $A_{3}$ is the best alternative.

Moreover, if we take $p=q=2$, then

$$
\begin{gathered}
h_{1}=\{0.1582,0.1612, \ldots, 0.2156\}, h_{2}=\{0.1460,0.1553, \ldots, 0.1935\} \\
h_{3}=\{0.1886,0.2015, \ldots, 0.1812\}
\end{gathered}
$$

and calculate the scores of all the alternatives, thus we get

$$
s\left(h_{1}\right)=0.1946, s\left(h_{2}\right)=0.1748, s\left(h_{3}\right)=0.1880
$$

Since $s\left(h_{3}\right)>s\left(h_{1}\right)>s\left(h_{2}\right)$, then the ranking of the alternatives $A_{i}(i=1,2,3)$ is $A_{1} \succ A_{3} \succ A_{2}$. Thus, the optimal alternative is $A_{1}$ in this case.

Therefore, the ranking results also depend on the values of the parameters $p$ and $q$ to some degree, and the proper values of the parameters should also be taken into account. Since the HFBM reduces to some other aggregation operators, such as the GHFM, the HFSM and the HFA operator, when $q$ approaches zero, it can only reflect "parting satisfaction" relationship of the aggregated arguments in such cases. We generally take the values of the two parameters as $p=q=1$ in practice, which can capture the interrelationship between the aggregated arguments as much as possible.

### 1.4 Hesitant Fuzzy Geometric Bonferroni Means

As an extension of the geometric mean (GM), the geometric Bonferroni mean (GBM) is a very useful aggregation operator, which considers the interrelationships among arguments. Xia et al. (2013b) introduced the concept of GBM:

Definition 1.27 (Xia et al. 2013b). Let $p, q>0$, and $a_{i}(i=1,2, \ldots, n)$ be a collection of non-negative numbers. If

$$
\begin{equation*}
G B^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\frac{1}{p+q} \prod_{\substack{i, j=1 \\ i \neq j}}^{n}\left(p a_{i}+q a_{j}\right)^{\frac{1}{n(n-1)}} \tag{1.101}
\end{equation*}
$$

then $G B^{p, q}$ is called a geometric Bonferroni mean (BGM).
In MADM, the performance of an alternative under an attribute may be represented by several possible values represented by a HFE. To aggregate all the performances of an alternative under all the attributes with connections among them, Zhu et al. (2012b) gave an extension of the GBM, which was defined as follows:

Definition 1.28 (Zhu et al. 2012b). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, if
then $H F G B^{p, q}$ is called a hesitant fuzzy geometric Bonferroni mean (HFGBM).
Based on the operational laws of the HFEs, we further develop the theorem related to the HFGBM as follows:

Theorem 1.32 (Zhu et al. 2012b). Let $p, q>0$, and $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, then the aggregated value by using the HFGBM is a HFE, and

$$
\begin{equation*}
\operatorname{HFGB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, j<j}}\left\{1-\left(1-\prod_{\substack{i, j=1 \\ i<j}}^{n}\left(\varepsilon_{i, j}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\} \tag{1.103}
\end{equation*}
$$

where $\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)$ can be considered as the "bonding satisfaction" factor used as a calculation unit, capturing the connection between $h_{i}$ and $h_{j}, i, j=1,2, \ldots, n, i \neq j$.

Proof. By the operational laws of HFEs, we obtain

$$
\begin{equation*}
\tau_{i, j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right) \tag{1.104}
\end{equation*}
$$

which is also a HFE, and

$$
\operatorname{HFGB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{p+q}\left({\underset{c}{n} \begin{array}{|c}
\underset{i}{i, j=1}  \tag{1.105}\\
i<j \\
\hline
\end{array}}\left(\tau_{i, j}\right)^{\frac{2}{n(n-1)}}\right)
$$

Furthermore, we have

$$
\begin{align*}
\bigotimes_{\substack{i, j=1 \\
i<j}}^{n}\left(\tau_{i, j}\right)^{\frac{2}{n(n-1)}} & =\left(\bigotimes_{\substack{i, j=1 \\
i<j}}^{n}\left(\tau_{i, j}\right)\right)^{\frac{2}{n(n-1)}}=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, i<j .}}\left\{\left(\prod_{i, j=1}^{n} \varepsilon_{i, j}\right)^{\frac{2}{n(n-1)}}\right\} \\
& =\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, i<j .}}\left\{\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\varepsilon_{i, j}\right)^{\frac{2}{n(n-1)}}\right\} \tag{1.106}
\end{align*}
$$

and then

$$
\begin{equation*}
\frac{1}{p+q} \bigotimes_{\substack{i, j=1 \\ i<j}}^{n}\left(\tau_{i, j}\right)^{\frac{2}{n(n-1)}}=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, i<j .}}\left\{1-\left(1-\left(\prod_{\substack{i, j=1 \\ i<j}}^{n}\left(\varepsilon_{i, j}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \tag{1.107}
\end{equation*}
$$

thus the proof is completed.
We can see that $\tau_{i, j}$ is a basic element, which Zhu et al. (2012b) called the hesitant fuzzy geometric Bonferroni element (HFGBE). The reasons they defined the HFGBE are that: for one it can fully represent the connection between two HFEs by two types of conjunction calculation; Second, after the original data being operated by two types of conjunction calculations, the values and quantity of them have been changed, and these changed basic arguments also change the performance of the aggregation operator. In the MADM problems, the HFGBE can reflect the advantage of HFGBM considering two factors in the aggregation process, which can take much more information into account.

We treat the HFGBE as the "bonding satisfaction" factor, which is used to define the HFGBM. If we utilize the method we define this factor to other aggregation operators, they should perform differently from the original ones.

Thus, in order to investigate the desirable properties of the HFGBM, we first discuss the HFGBE in detail.

Theorem 1.33 (Zhu et al. 2012b). Let $h_{\alpha_{i}}$ and $h_{\beta_{i}}$ be two collections of HFEs,

$$
\begin{align*}
\tau_{\alpha_{i, j, i<j}} & =\left(p h_{\alpha_{i}} \oplus q h_{\alpha_{j}}\right) \otimes\left(p h_{\alpha_{j}} \oplus q h_{\alpha_{i}}\right)  \tag{1.108}\\
\tau_{\beta_{i, j, k j}} & =\left(p h_{\beta_{i}} \oplus q h_{\beta_{j}}\right) \otimes\left(p h_{\beta_{j}} \oplus q h_{\beta_{i}}\right) \tag{1.109}
\end{align*}
$$

if for any $\quad \gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \quad \gamma_{\beta_{i}} \in h_{\beta_{i}}, \quad$ we have $\quad \gamma_{\alpha_{i}} \leq \gamma_{\beta_{i}}, \quad \gamma_{\alpha_{j}} \leq \gamma_{\beta_{j}}$, $i, j=1,2, \ldots, n, i \neq j$, then $\tau_{\alpha_{i, j, i<j}} \leq \tau_{\beta_{i, j, i<j}}$.

Proof. Since $\gamma_{\alpha_{i}} \leq \gamma_{\beta_{i}}$ and $\gamma_{\alpha_{j}} \leq \gamma_{\beta_{j}}, i, j=1,2, \ldots, n, i \neq j$, then we have

$$
\begin{align*}
& 1-\left(1-\gamma_{\alpha_{i}}\right)^{p}\left(1-\gamma_{\alpha_{j}}\right)^{q} \leq 1-\left(1-\gamma_{\beta_{i}}\right)^{p}\left(1-\gamma_{\beta_{j}}\right)^{q}  \tag{1.110}\\
& 1-\left(1-\gamma_{\alpha_{j}}\right)^{p}\left(1-\gamma_{\alpha_{i}}\right)^{q} \leq 1-\left(1-\gamma_{\beta_{j}}\right)^{p}\left(1-\gamma_{\beta_{i}}\right)^{q} \tag{1.111}
\end{align*}
$$

Additionally, we obtain

$$
\begin{aligned}
& \tau_{\alpha_{i, j, i<j}}=\left(p h_{\alpha_{i}} \oplus q h_{\alpha_{j}}\right) \otimes\left(p h_{\alpha_{j}} \oplus q h_{\alpha_{i}}\right) \\
& =\left(\underset{\gamma_{\alpha_{i}} \in h_{\alpha_{i},}, \gamma_{\alpha_{j}} \in h_{\alpha_{j}}}{\bigcup}\left\{1-\left(1-\gamma_{\alpha_{i}}\right)^{p}+1-\left(1-\gamma_{\alpha_{j}}\right)^{q}-\left(1-\left(1-\gamma_{\alpha_{i}}\right)^{p}\right)\left(1-\left(1-\gamma_{\alpha_{j}}\right)^{q}\right)\right\}\right) \\
& \otimes\left(\bigcup_{\gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \gamma_{\alpha_{j}} \in h_{\alpha_{j}}}^{\bigcup}\left\{1-\left(1-\gamma_{\alpha_{j}}\right)^{p}+1-\left(1-\gamma_{\alpha_{i}}\right)^{q}-\left(1-\left(1-\gamma_{\alpha_{j}}\right)^{p}\right)\left(1-\left(1-\gamma_{\alpha_{i}}\right)^{q}\right)\right\}\right) \\
& =\left(\bigcup_{\gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \gamma_{\alpha_{j}} \in h_{\alpha_{j}}}^{\bigcup}\left\{1-\left(1-\gamma_{\alpha_{i}}\right)^{p}\left(1-\gamma_{\alpha_{j}}\right)^{q}\right\}\right) \otimes\left(\bigcup_{\gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \gamma_{\alpha_{j}} \in h_{\alpha_{j}}}\left\{1-\left(1-\gamma_{\alpha_{j}}\right)^{p}\left(1-\gamma_{\alpha_{i}}\right)^{q}\right\}\right)
\end{aligned}
$$

Let $\varepsilon_{\alpha_{i, j, i<j}} \in \tau_{\alpha_{i, j, i<j}}, \varepsilon_{\beta_{i, j, i<j}} \in \tau_{\beta_{i, j, i<j}}$, for all $i, j=1,2, \ldots, n, i<j$, we have $\varepsilon_{\alpha_{i, j, i<j}} \leq \varepsilon_{\beta_{i, j, j<j}}$ and $\tau_{\alpha_{i, j, i<j}} \leq \tau_{\beta_{i, j, i<j}}$, which completes the proof.

Theorem 1.34 (Zhu et al. 2012b). Let $h_{\alpha_{i}}$ and $h_{\beta_{i}}$ be two collections of HFEs, $\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right), \quad h_{i}^{-}=\bigcup_{\gamma_{i} \in h_{i}} \min \left\{\gamma_{i}\right\}, \quad h_{i}^{+}=\bigcup_{\gamma_{i} \in h_{i}} \max \left\{\gamma_{i}\right\}$, $\gamma^{-} \in h_{i}^{-}, \gamma^{+} \in h_{i}^{+}, i, j=1,2, \ldots, n, i \neq j$, then

$$
\begin{equation*}
\bigcup_{\gamma^{-} \in h_{i}^{-}}\left(1-\left(1-\gamma^{-}\right)^{p+q}\right)^{2} \leq \tau_{i, j, i<j} \leq \bigcup_{\gamma^{\prime} \in h_{i}^{+}}\left(1-\left(1-\gamma^{+}\right)^{p+q}\right)^{2} \tag{1.112}
\end{equation*}
$$

Proof. Since $\gamma^{-} \leq \gamma_{i} \leq \gamma^{+}(i=1,2, \ldots, n)$, then

$$
\begin{align*}
& 1-\left(1-\gamma^{-}\right)^{p+q} \leq 1-\left(1-\gamma_{i}\right)^{p}\left(1-\gamma_{j}\right)^{q} \leq 1-\left(1-\gamma^{+}\right)^{p+q}  \tag{1.113}\\
& 1-\left(1-\gamma^{-}\right)^{p+q} \leq 1-\left(1-\gamma_{j}\right)^{p}\left(1-\gamma_{i}\right)^{q} \leq 1-\left(1-\gamma^{+}\right)^{p+q} \tag{1.114}
\end{align*}
$$

and

$$
\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)
$$

$$
\begin{equation*}
=\left(\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \epsilon_{j}}\left\{1-\left(1-\gamma_{i}\right)^{p}\left(1-\gamma_{j}\right)^{q}\right\}\right) \otimes\left(\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{1-\left(1-\gamma_{j}\right)^{p}\left(1-\gamma_{i}\right)^{q}\right\}\right) \tag{1.115}
\end{equation*}
$$

which completes the proof.

Theorem 1.35 (Zhu et al. 2012b). When we exchange the parameters $p$ and $q$, we obtain

$$
\begin{align*}
\tau_{i, j, i<j} & =\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right) \\
& =\left(q h_{i} \oplus p h_{j}\right) \otimes\left(q h_{j} \oplus p h_{i}\right)=\tau_{i, j, i<j} \tag{1.116}
\end{align*}
$$

which is the property of "idempotent commutativity" (Xu and Da 2012b).
Theorem 1.36 (Zhu et al. 2012b). When we take $h=h_{i}=h_{j}=\{0\}$, or $h=h_{i}=h_{j}=\{1\}$, respectively, the corresponding results are valid:

$$
\begin{equation*}
\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)=\{0\} \tag{1.117}
\end{equation*}
$$

or

$$
\begin{equation*}
\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)=\{1\} \tag{1.118}
\end{equation*}
$$

Based on the study above, we can further investigate some desirable properties of the HFGBM as follows:

Theorem 1.37 (Zhu et al. 2012b). Let $\left(h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right)$ and $\left(h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right)$ be two collections of HFEs, if for any $\gamma_{\alpha_{i}} \in h_{\alpha_{i}}, \gamma_{\beta_{i}} \in h_{\beta_{i}}$, we have $\gamma_{\alpha_{i}} \leq \gamma_{\beta_{i}}$ for all $i, j=1,2, \ldots, n, i \neq j$, then

$$
\begin{equation*}
H F G B^{p, q}\left(h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right) \leq H F G B^{p, q}\left(h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right) \tag{1.119}
\end{equation*}
$$

Proof. Since

$$
\begin{equation*}
1-\left(1-\left(\prod_{i, j=1}^{n}\left(\varepsilon_{\alpha_{i, j}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}} \leq 1-\left(1-\left(\prod_{\substack{i, j=1 \\ i<j}}^{n}\left(\varepsilon_{\beta_{i, j}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}} \tag{1.120}
\end{equation*}
$$

then

$$
\begin{align*}
& \frac{1}{p+q} \otimes_{\substack{i, j=1 \\
i<j}}^{n}\left(\tau_{\alpha_{i, j}}\right)^{\frac{2}{n(n-1)}}=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, i<j}}\left\{1-\left(1-\left(\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\varepsilon_{\alpha_{i, j}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \\
& \quad \leq \frac{1}{p+q} \bigotimes_{\substack{i, j=1 \\
i<j}}^{n}\left(\tau_{\beta_{i, j}}\right)^{\frac{2}{n(n-1)}}=\bigcup_{\varepsilon_{i, j} \in \tau_{i, j, i<j}}\left\{1-\left(1-\left(\prod_{i, j=1}^{n}\left(\varepsilon_{\beta_{i, j}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \tag{1.121}
\end{align*}
$$

which completes the proof of the theorem.

Theorem 1.38 (Zhu et al. 2012b). Let $\left(h_{\alpha_{1}}, h_{\alpha_{2}}, \ldots, h_{\alpha_{n}}\right)$ and $\left(h_{\beta_{1}}, h_{\beta_{2}}, \ldots, h_{\beta_{n}}\right)$ be two collections of HFEs, $h_{i}^{-}=\bigcup_{\gamma_{i} \in h_{i}} \min \left\{\gamma_{i}\right\}$, $h_{i}^{+}=\bigcup_{\gamma_{i} \in h_{i}} \max \left\{\gamma_{i}\right\}, \gamma^{-} \in h_{i}^{-}, \gamma^{+} \in h_{i}^{+}, i, j=1,2, \ldots, n, i \neq j$, then

$$
\begin{gather*}
\bigcup_{\gamma^{-} \in h_{i}^{-}}\left\{1-\left(1-\left(1-\left(1-\gamma^{-}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}}\right\} \leq H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \\
\leq \bigcup_{\gamma^{+} \in h_{i}^{+}}\left\{1-\left(1-\left(1-\left(1-\gamma^{+}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}}\right\} \tag{1.122}
\end{gather*}
$$

Proof. We have

$$
\begin{align*}
1-\left(1-\left(1-\left(1-\gamma^{-}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}} & \leq 1-\left(1-\left(\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\varepsilon_{i, j}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}} \\
& \leq 1-\left(1-\left(1-\left(1-\gamma^{+}\right)^{p+q}\right)^{2}\right)^{\frac{1}{p+q}} \tag{1.123}
\end{align*}
$$

which completes the proof of the theorem.

Theorem 1.39 (Zhu et al. 2012b). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, and $\left(\dot{h}_{1}, \dot{h}_{2}, \ldots, \dot{h}_{n}\right)$ any permutation of $\left(h_{1}, h_{2}, \ldots, h_{n}\right)$. Apparently, we have

$$
\begin{align*}
\operatorname{HFGB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)= & \left.\frac{1}{p+q}{\stackrel{\bigotimes}{\substack{i, j=1 \\
i<j}}{ }_{\bigotimes}^{n}\left(\tau_{i, j}\right)^{\frac{2}{n(n-1)}}}=\frac{1}{p+q_{i, j}^{i, j=j}} \stackrel{\bigotimes}{i=j}_{n}^{\left(\dot{\tau}_{i j}\right.}\right)^{\frac{2}{n(n-1)}}=\operatorname{HFGB}^{p, q}\left(\dot{h}_{1}, \dot{h}_{2}, \ldots, \dot{h}_{n}\right)
\end{align*}
$$

where $\tau_{i, j, i<j}=\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right), \dot{\tau}_{i, j, i<j}=\left(p \dot{h}_{i} \oplus q \dot{h}_{j}\right) \otimes\left(p \dot{h}_{j} \oplus q \dot{h_{i}}\right)$, $i, j=1,2, \ldots, n, i \neq j$.

Theorem 1.40 (Zhu et al. 2012b). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, then

$$
\begin{aligned}
& H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{p+q}{\underset{Q}{i, j=1}}_{\substack{i, j \\
i \neq j}}\left(\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)\right)^{\frac{2}{n(n-1)}}
\end{aligned}
$$

Theorem 1.41 (Zhu et al. 2012b). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, if $h=h_{i}=\{0\}$, then we have

$$
\begin{equation*}
H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\{0\} \tag{1.126}
\end{equation*}
$$

and if $h=h_{i}=\{1\}$, then we get

$$
\begin{equation*}
H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\{1\} \tag{1.127}
\end{equation*}
$$

Next, we change the parameters $p$ and $q$ of the HFGBM, and we can get some special cases as follows (Zhu et al. 2012b):

Case 1. If $q \rightarrow 0$, then we have

$$
\begin{align*}
& =\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{1-\left(1-\prod_{\substack{i=1, j \\
i \neq j}}^{n}\left(\left(1-\left(1-\gamma_{i}\right)^{p}\right)\left(1-\left(1-\gamma_{j}\right)^{p}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p}}\right\} \\
& =H F G B^{p, 0}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \tag{1.128}
\end{align*}
$$

which we call a generalized hesitant fuzzy geometric Bonferroni mean (GHFGBM).

Case 2. If $p=1$ and $q \rightarrow 0$, then

$$
\begin{align*}
& \operatorname{HFGB} B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigotimes_{\substack{i, j=1 \\
i \neq j}}^{n}\left(h_{i} \otimes h_{j}\right)^{\frac{1}{n(n-1)}} \\
&=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{\prod_{\substack{i=1, j \\
i \neq j}}^{n}\left(\left(1-\left(1-\gamma_{i}\right)\right)\left(1-\left(1-\gamma_{j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right\} \tag{1.129}
\end{align*}
$$

which we call a hesitant fuzzy geometric Bonferroni mean (HFGBM).
Case 3. If $p=2$ and $q \rightarrow 0$, then

$$
H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{2}\left({\left.\left.\underset{\substack{i, j=1 \\ i \neq j}}{n}\left(2 h_{i} \otimes 2 h_{j}\right)^{\frac{1}{n(n-1)}}\right)\right) ~}_{\substack{ \\i}}\right.
$$

$$
\begin{gather*}
=\bigcup_{\gamma_{i} \in h_{i}, \gamma_{j} \in h_{j}}\left\{1-\left(1-\prod_{\substack{i=1, j \\
i \neq j}}^{n}\left(\left(1-\left(1-\gamma_{i}\right)^{2}\right)\left(1-\left(1-\gamma_{j}\right)^{2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right\} \\
=H F G B^{2,0}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \tag{1.130}
\end{gather*}
$$

which we call a hesitant fuzzy square geometric Bonferroni mean (HFSGBM).
Case 4. If $p=q=1$, and let

$$
\begin{equation*}
\tau_{i, j}^{1,1}=\left(h_{i} \oplus h_{j}\right) \otimes\left(h_{j} \oplus h_{i}\right)=\bigcup_{\varepsilon_{i, j}^{1, j} \in \tau_{i, j}^{1, j}}\left\{\varepsilon_{i, j}^{1,1}\right\} \tag{1.131}
\end{equation*}
$$

then

$$
\begin{align*}
& \operatorname{HFGB}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{2}\left({\left.\underset{\substack{i, j=1 \\
i \neq j}}{n}\left(\left(h_{i} \oplus h_{j}\right) \otimes\left(h_{j} \oplus h_{i}\right)\right)^{\frac{2}{n(n-1)}}\right)}_{=\bigcup_{\mathcal{E}_{i, j}^{1,1} \in \tau_{i, j, i<j}}\left\{1-\left(1-\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\varepsilon_{i, j}^{1,1}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\right\}=H F G B^{1,1}\left(h_{1}, h_{2}, \ldots, h_{n}\right)}\right.
\end{align*}
$$

which we call a hesitant fuzzy interrelated square geometric Bonferroni mean (HFISGBM).

Now, we give an example to illustrate the results above:
Example 1.18 (Zhu et al. 2012b). Assume that we have three HFEs, $h_{1}=\{0.1\}$, $h_{2}=\{0.2,0.4\}$ and $h_{3}=\{0.3\}$. Then based on the operation laws of HFEs in Definition 1.7, we have

$$
\begin{aligned}
& h_{1} \oplus h_{2}=\{0.28,0.46\}, h_{2} \oplus h_{1}=\{0.28,0.46\}, h_{1} \oplus h_{3}=\{0.37\} \\
& h_{3} \oplus h_{1}=\{0.37\}, h_{2} \oplus h_{3}=\{0.44,0.58\}, h_{3} \oplus h_{2}=\{0.44,0.58\}
\end{aligned}
$$

and

$$
\begin{gathered}
H F G B^{1,1}\left(h_{1}, h_{2}, h_{3}\right)=\{0.0661,0.0784,0.0726,0.0863,0.0932, \\
0.1027,0.0799,0.0950,0.1132\} \\
H F G B^{2,2}\left(h_{1}, h_{2}, h_{3}\right)=\{0.0991,0.1156,0.1065,0.1245,0.1355, \\
0.1464,0.1146,0.1343,0.1584\} \\
H F G B^{1,0}\left(h_{1}, h_{2}, h_{3}\right)=\{0.0330,0.0416,0.0524\} \\
H F G B^{0,1}\left(h_{1}, h_{2}, h_{3}\right)=\{0.0330,0.0416,0.0524\}
\end{gathered}
$$

and

$$
\begin{aligned}
& s\left(H F G B^{1,1}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.0872, s\left(\operatorname{HFGB}^{1,0}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.0422 \\
& s\left(H F G B^{2,2}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.1257, s\left(\operatorname{HFGB}^{0,1}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.0422
\end{aligned}
$$

When the values of the parameters $p$ and $q$ change, more details can be found in Figs. 1.5-1.7 (Zhu et al. 2012b):


Fig. 1.5. Scores of the HFGBM $(p=q, p \in(0,40])$


Fig. 1.6. Scores of the $\operatorname{HFGBM}(p=q, p \in(0,40])$


Fig. 1.7. Scores of the HFGBM $(q=0, p \in(0,40] ; p=0, q \in(0,40])$

If the interrelationships among arguments reflected by the BM are an impersonal character, then the correlations among arguments represented by Choquet integral are a personal character. We often have to think through them in practical situations, which is more comprehensive. Now we combine the HFGBM with Choquet integral. We know that the HFGBE is a basic element used as a calculation unit, and we still utilize it in the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGM).

Let $\stackrel{\sim i, j ; i<j}{v}$ be the $n-1$ tuple $\left\{\bar{\tau}_{\sigma_{(i, j, i<j)}(1)}, \bar{\tau}_{\sigma_{(i, j, i<j)}(2)}, \ldots, \bar{\tau}_{\sigma_{(i, j, i<j)}(n)}\right\}$ which doesn't have the element $\bar{\tau}_{i, j, i<j}$, and $\bar{\tau}_{\sigma_{(i, j, i<j)}(1)}, \bar{\tau}_{\sigma_{(i, j, i<j)}(2)}, \ldots, \bar{\tau}_{\sigma_{(i, j, i<j)}(n)}$ are the ordered interval numbers in $\stackrel{\sim i, j ; i<j}{v}$, such that $\bar{\tau}_{\sigma_{(i, j, i<j)}\left(l_{0}-1\right)} \geq \bar{\tau}_{\sigma_{(i, j, i<j)}\left(l_{0}\right)}$, $l_{0}=3,4, \ldots, n-2$. Let $\quad \dot{B}_{\sigma_{i, j, i<j}(k)}=\left\{\bar{\tau}_{\sigma_{(i, j, i<j)}\left(l_{0}\right)} \mid l_{0} \leq k\right\}$, when $k \geq 2$ and $\dot{B}_{\sigma_{i, j, i<j}(1)}=\phi$.

Let $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be a set of attributes, $X^{\sim i, j}=X-\left\{x_{i}, x_{j}\right\}$ the set of all attributes except $x_{i}$ and $x_{j}$. A fuzzy measure $\dot{m}_{i, j, i<j}$ on $X^{\sim i, j}$ is $m_{i, j, i<j}\left(X^{\sim i, j}\right) \rightarrow[0,1]$, satisfying the axioms below:
(1) $\dot{m}_{i, j, i<j}(\phi)=0$.
(2) $\dot{m}_{i, j, i<j}\left(X^{\sim i, j}\right)=1$.
(3) $\dot{m}_{i, j, i<j}\left(X_{1}\right) \leq \dot{m}_{i, j, i<j}\left(X_{2}\right)$, if $X_{1} \subseteq X_{2}$.

Let

$$
\begin{equation*}
\rho_{i, j, i<j}=\oplus_{\substack{i, j=1 \\ i \neq j}}^{n} \stackrel{\sim i, j ; i<j}{\mathcal{V}}\left(\dot{m}_{i, j, i<j}\left(\dot{B}_{\sigma_{i, j, i<j}(k)}\right)-\dot{m}_{i, j, i<j}\left(\dot{B}_{\sigma_{i, j, i<j}(k-1)}\right)\right), k=1,2, \ldots, n-2 \tag{1.133}
\end{equation*}
$$

then the Choquet integral of $\underset{v}{\sim i, j ; i<j}$ with respect to $\dot{m}_{i, j, i<j}$ can be defined as:

$$
\begin{equation*}
C_{\dot{m}_{i, j, i<j}}\binom{\sim i, j ; i<j}{\boldsymbol{V}}=\frac{1}{l_{\rho_{i, j, i<j}}} \sum_{\delta_{i, j, i<j} \in \rho_{i, j, i<j}} \delta_{i, j, i<j} \tag{1.134}
\end{equation*}
$$

where $i, j=1,2, \ldots, n, i \neq j ; k=1,2, \ldots, n-2$. Thus, we have

$$
\begin{align*}
& \operatorname{HFCGB}^{p, q}\left(\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right. \\
& =\frac{1}{p+q}\left({\left.\underset{\substack{i, j=1 \\
i \neq j}}{n}\left(\left(\left(p h_{i} \oplus q h_{j}\right) \otimes\left(p h_{j} \oplus q h_{i}\right)\right)^{C_{m_{i, j}, i<j}\binom{-i, j, i<j}{v}}\right)^{\frac{2}{n(n-1)}}\right)}^{n}\right) \tag{1.135}
\end{align*}
$$

where $p, q>0$, then $H F C G B^{p, q}$ is called a hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM), and

$$
\begin{align*}
& =\bigcup_{\varepsilon_{i, j} \in \bar{\tau}_{i, j, i<j}}\left\{1-\left(1-\left(\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\left(\varepsilon_{i, j}\right)^{c_{m_{i, j, i<j}}\binom{-i, j, j<j}{v}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \tag{1.136}
\end{align*}
$$

Example 1.19 (Zhu et al. 2012b). Assume the same data in Example 1.17, let $p=q=1$ and

$$
\begin{gathered}
\dot{m}_{1,2}(\phi)=\dot{m}_{2,3}(\phi)=\dot{m}_{1,3}(\phi)=0, \dot{m}_{1,3}\left(\left\{h_{2,3}\right\}\right)=0.4 \\
\dot{m}_{1,2}\left(\left\{h_{2,3}\right\}\right)=0.3 \\
\dot{m}_{1,2}\left(\left\{h_{1,3}\right\}\right)=\dot{m}_{1,3}\left(\left\{h_{1,2}\right\}\right)=0.2, \dot{m}_{2,3}\left(\left\{h_{1,2}\right\}\right)=\dot{m}_{2,3}\left(\left\{h_{1,3}\right\}\right)=0.5 \\
\dot{m}_{1,2}\left(\left\{h_{2,3}, h_{1,3}\right\}\right)=\dot{m}_{1,3}\left(\left\{h_{2,3}, h_{1,2}\right\}\right)=\dot{m}_{2,3}\left(\left\{h_{1,2}, h_{1,3}\right\}\right)=1
\end{gathered}
$$

then we have

$$
s\left(\bar{\tau}_{12}\right)=0.0201, s\left(\bar{\tau}_{13}\right)=0.0365, s\left(\bar{\tau}_{23}\right)=0.0744
$$

and thus $s\left(\bar{\tau}_{23}\right)>s\left(\bar{\tau}_{13}\right)>s\left(\bar{\tau}_{12}\right)$.

$$
\begin{aligned}
\rho_{1,2} & =\bar{\tau}_{2,3}\left(\dot{m}_{1,2}\left(\left\{h_{2,3}\right\}\right)-\dot{m}_{1,2}(\phi)\right) \oplus \bar{\tau}_{1,3}\left(\dot{m}_{1,2}\left(\left\{h_{1,3}, h_{2,3}\right\}\right)-\dot{m}_{1,2}\left(\left\{h_{2,3}\right\}\right)\right) \\
& =0.3 \bar{\tau}_{2,3} \oplus 0.7 \bar{\tau}_{1,3} \\
\rho_{2,3} & =\bar{\tau}_{1,3}\left(\dot{m}_{2,3}\left(\left\{h_{1,2}\right\}\right)-\dot{m}_{1,2}(\phi)\right) \oplus \bar{\tau}_{1,2}\left(\dot{m}_{2,3}\left(\left\{h_{1,3}, h_{1,2}\right\}\right)-\dot{m}_{1,2}\left(\left\{h_{1,2}\right\}\right)\right) \\
& =0.5 \bar{\tau}_{1,3} \oplus 0.5 \bar{\tau}_{1,2} \\
\rho_{1,3} & =\bar{\tau}_{2,3}\left(m_{1,3}\left(\left\{h_{2,3}\right\}\right)-\dot{m}_{1,3}(\phi)\right) \oplus \bar{\tau}_{1,2}\left(\dot{m}_{1,3}\left(\left\{h_{1,2}, h_{2,3}\right\}\right)-\dot{m}_{1,3}\left(\left\{h_{2,3}\right\}\right)\right) \\
& =0.4 \bar{\tau}_{2,3} \oplus 0.6 \bar{\tau}_{1,2} \\
& C_{\dot{m}_{1,2}}\binom{\sim 1,2}{v}=0.1769, C_{\dot{m}_{1,3}}\binom{\sim 1,3}{v}=0.0062, C_{\dot{m}_{2,3}}\binom{\sim 2,3}{v}=0.0263
\end{aligned}
$$

Hence, we obtain
$\left.\left.\left.H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{p+q_{i}^{i, j=1}} \stackrel{\bigotimes}{i<j}_{n}^{( }\right)\left(\bar{\tau}_{i, j}\right)\right)^{\left.c_{m_{i, j, i<j}(i, j, j i<j}\right)}\right)^{\frac{2}{n(n-1)}}$
$=\{0.6061,0.6087,0.6113,0.6393,0.6423,0.6452,0.6771,0.6805,0.6839\}$

In practical situation, we have to deal with the complicated situations, not only considering the importance of individual arguments but also the relations among them. In order to fully consider the connections among attributes in the MADM problems, Zhu et al. (2012b) introduced the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and the weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM) below:

Definition 1.29 (Zhu et al. 2012b). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$ the weight vector of $h_{i}$, where $w_{i}$ indicates the importance degree of $h_{i}$, satisfying $w_{i}>0, i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$. If
$H F G B^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\frac{1}{p+q}\left(\underset{\substack{i, j=1 \\ i \neq j}}{n}\left(\left(p h_{i}^{w_{i}} \oplus q h_{j}^{w_{j}}\right) \otimes\left(p h_{j}^{w_{j}} \oplus q h_{i}^{w_{i}}\right)\right)^{\frac{2}{n(n-1)}}\right)$
where $p, q>0$, then $H F G B_{w}^{p, q}$ is called a weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM).

Let

$$
\begin{equation*}
\bar{\tau}_{i, j, i<j}^{w}=\left(p h_{i}^{w_{i}} \oplus q h_{j}^{w_{j}}\right) \otimes\left(p h_{j}^{w_{j}} \oplus q h_{i}^{w_{i}}\right) \tag{1.138}
\end{equation*}
$$

then we have

Theorem 1.42 (Zhu et al. 2012b). Let $h_{i}(i=1,2, \ldots, n)$ be a collection of HFEs, whose weight vector is $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$, which satisfies $w_{i}>0$, $i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$. Then the aggregated value by using the WHFGBM is also a HFE, and

$$
\begin{align*}
\operatorname{HFGB}_{w}^{p, q}\left(h_{1}, h_{2}, \ldots, h_{n}\right) & =\frac{1}{p+q} \stackrel{\otimes}{i}, j=1_{i, j}^{i<j}\left(\bar{\tau}_{i, j}^{w}\right)^{\frac{2}{n(n-1)}} \\
& =\bigcup_{\varepsilon_{i, j}^{w} \in \tau_{i, j, i<j}^{w}}\left\{1-\left(1-\left(\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\varepsilon_{i, j}^{w}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \tag{1.139}
\end{align*}
$$

Next, we introduce an approach for MADM under hesitant fuzzy environment below (Zhu et al. 2012b):

Step 1. For a MADM problem, let $A_{i}(i=1,2, \ldots, n)$ be a collection of $n$ alternatives, $x_{j}(j=1,2, \ldots, m)$ a collection of $m$ attributes, whose weight vector is $w=\left(w_{1}, w_{2}, \ldots, w_{m}\right)^{\mathrm{T}}$, satisfying $w_{j}>0, j=1,2, \ldots, m$, and
$\sum_{j=1}^{m} w_{j}=1$, where $w_{j}$ denotes the importance degree of the attribute $x_{j}$. The DMs provide all the possible values that the alternative $A_{i}$ satisfy the attribute $x_{j}$ represented by the HFEs $h_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\gamma_{i j}\right\}(i=1,2, \ldots, n ; j=1,2, \ldots, m)$, which are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{n \times m}$.

Then we may transform the decision matrix $H=\left(h_{i j}\right)_{n \times m}$ into the normalization matrix $B=\left(b_{i j}\right)_{n \times m}$ by using the formula (1.52).

Step 2. Utilize the WHFGBM (in general, we can take $p \neq 0$ and $q \neq 0$ ) to aggregate all the performance values $b_{i j}(j=1,2, \ldots, n)$ of the $i$ th line and get the overall performance value $b_{i}$ corresponding to the alternative $A_{i}: b_{i}=$ $W H F B_{w}^{p, q}\left(b_{i 1}, b_{i 2}, \ldots, b_{i m}\right)$.

Step 3. Calculate the scores $s\left(b_{i}\right)(i=1,2, \ldots, n)$ of $b_{i}(i=1,2, \ldots, n)$ and rank all the alternatives $A_{i}(i=1,2, \ldots, n)$ according to $s\left(b_{i}\right)(i=1,2, \ldots, n)$ in descending order.

In the following, we apply the given method to a MADM problem:
Example 1.20 (Zhu et al. 2012b). Let us consider a factory which intends to select a new site for new buildings. Three kinds of alternatives $A_{i}(i=1,2,3)$ are available. The DMs consider three attributes to decide which site to choose: $x_{1}$ (price), $x_{2}$ (location), $x_{3}$ (environment). The weight vector of the attributes $x_{j}(j=1,2,3)$ is $w=(0.5,0.3,0.2)^{\mathrm{T}}$. Assume that the characteristics of the alternatives $A_{i}(i=1,2,3)$ with respect to the attributes $x_{j}(j=1,2,3)$ are represented by the HFEs $h_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\gamma_{i j}\right\}$, where $\gamma_{i j}$ indicates the degree that the alternative $A_{i}$ satisfies the attribute $x_{j}$. All $h_{i j}(i=1,2,3 ; j=1,2,3)$ are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{3 \times 3}$ (see Table 1.7 (Zhu et al. 2012b)).

Table 1.7. The hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.6,0.7,0.8\}$ | $\{0.25\}$ | $\{0.4,0.5\}$ |
| $A_{2}$ | $\{0.4\}$ | $\{0.4,0.5\}$ | $\{0.3,0.55,0.6\}$ |
| $A_{3}$ | $\{0.2,0.4\}$ | $\{0.6,0.5\}$ | $\{0.7,0.5\}$ |

Considering that all the attributes $x_{j}(j=1,2,3)$ are the benefit type attributes, the performance values of the alternatives $A_{i}(i=1,2,3)$ do not need normalization. We first utilize the WHFGBM (here, we take $p=q=1$ ) to aggregate all the performance values $h_{i j}(j=1,2,3)$ of the $i$ th line and get the overall performance value $h_{i}$ corresponding to the alternative $A_{i}$, then we can calculate the scores of all the alternatives:

$$
s\left(h_{1}\right)=0.7076, s\left(h_{2}\right)=0.6766, s\left(h_{3}\right)=0.6997
$$

Since $s\left(h_{1}\right)>s\left(h_{3}\right)>s\left(h_{2}\right)$, then the ranking of the HFEs: $h_{1}>h_{3}>h_{2}$, and thus, the ranking of the alternatives $A_{i}(i=1,2,3)$ is $A_{1} \succ A_{3} \succ A_{2}$. Hence, $A_{1}$ is the best alternative.

If we take $p=3$ and $q=1$, then we can calculate the scores of all the alternatives:

$$
s\left(h_{1}\right)=0.7270, s\left(h_{2}\right)=0.6979, s\left(h_{3}\right)=0.6825
$$

Since $s\left(h_{1}\right)>s\left(h_{2}\right)>s\left(h_{3}\right)$, then we can get the ranking of the HFEs, that is $h_{1}>h_{2}>h_{3}$, and thus, the ranking of the alternatives $A_{i}(i=1,2,3)$ is $A_{1} \succ A_{2} \succ A_{3}$. Hence, $A_{2}$ is the best choice.

Zhu et al. (2013a) introduced the WHFBM as listed in Eq.(1.98) or (1.99), now, we use Eq.(1.98) to aggregate the same data, we obtain the scores of all the alternatives as below:

$$
s\left(h_{1}\right)=0.1820, s\left(h_{2}\right)=0.1703, s\left(h_{3}\right)=0.1828
$$

Then we give the ranking of $A_{i}(i=1,2,3)$ as $A_{3} \succ A_{1} \succ A_{2}$.

Comparing the two ranking results, we can find a change in sequence happening among alternatives. That's because the WHFGBM operator pays more attention to some arguments, whose performances are too high or too low, however, the WHFBM operator focuses on the whole arguments instead. So, the attribute values $h_{11}=\{0.6,0.7,0.8\}$ of the alternative $A_{1}$ affect the ranking in the end.

Based on the above results, we know that the WHFBM and the WHFGBM have different emphases in the case of aggregating arguments, and when we change the values of the parameters $p$ and $q$, we also can get different rankings of the alternatives. In practical applications, we may use different operators and control parameters to deal with different situations according to the realism.

Moreover, in practical applications, the preferences of the DMs should also be taken into account, and Choquet integral can deal with this situation. Thus we not only consider the importance of each attribute but also the correlations of attributes. Now, we combine the WHFGBM with Choquet integral as follows:

Let $\stackrel{\sim i, j ; i<j}{v_{w}}$ be the $n-1$ tuple $\left\{\bar{\tau}_{\sigma_{(i, j, i<j)}}^{w}, \bar{\tau}_{\sigma_{(i, j, i, j)}(2)}^{w}, \ldots, \bar{\tau}_{\sigma_{(i, j, i<j)}(n)}^{w}\right\}$ which doesn't have the element $\bar{\tau}_{i, j, i<j}^{w}$, and $\bar{\tau}_{\sigma_{(i, j, i j)}(1)}^{w}, \bar{\tau}_{\sigma_{(i, j, i<j)}(2)}^{w}, \ldots, \bar{\tau}_{\sigma_{(i, j, i<j)}(n)}$ are the ordered interval numbers in $\stackrel{\sim}{c}_{v_{w}}^{\sim i, j ; i<j}$, such that $\bar{\tau}_{\sigma_{(i, j, i<j)}\left(l_{0}-1\right)}^{w} \geq \bar{\tau}_{\sigma_{(i, j, i<j)}\left(l_{0}\right)}^{w}$, $l_{0}=3,4, \ldots, n-2 ; \quad$ Let $\left.\quad \dot{B}_{\sigma_{i, j, i<j}(k)}^{w}=\left\{\bar{\tau}_{\sigma_{(i, j, i<j)}}^{w} \mid l_{0}\right) \mid l_{0} \leq k\right\}, \quad$ when $\quad k \geq 2 \quad$ and $\dot{B}_{\sigma_{i, j, i<j(1)}}^{w}=\phi . \dot{m}_{i, j, i<j}$ is a fuzzy measure on $X^{\sim i, j}$ defined previously. Let
$\rho_{i, j, i<j}^{w}=\stackrel{n}{\substack{i, j=1 \\ i \neq j}}\left(\left(p h_{i}^{w_{i}} \oplus q h_{j}^{w_{j}}\right) \otimes\left(p h_{j}^{w_{j}} \oplus q h_{i}^{w_{i}}\right)\right)\left(\dot{m}_{i, j, i<j}\left(\dot{B}_{\sigma_{i, j, k j}}^{w}(k)\right)-\dot{m}_{i, j, i<j}\left(\dot{B}_{\sigma_{i, j, i<j}(k-1)}^{w}\right)\right)$
where $k=1,2, \ldots, n-2$, then Choquet integral of $\stackrel{\sim i, j ; i<j}{v_{w}}$ with respect to $\dot{m}_{i, j, i<j}$ can be defined as:

$$
\begin{equation*}
C_{\dot{m}_{i, j, i<j}}^{w}\binom{\sim i, j ; i<j}{V_{w}}=\frac{1}{l_{\rho_{i, j, i<j}^{w}}} \sum_{\delta_{i, j, i<j}^{w} \in \rho_{i, j, i<j}^{w}} \delta_{i, j, i<j}^{w} \tag{1.141}
\end{equation*}
$$

where $i, j=1,2, \ldots, n, i \neq j ; k=1,2, \ldots, n-2$.

Hence, we have

$$
\begin{align*}
& H F C G B_{w}^{p, q}\left(\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right. \\
& \left.\quad=\frac{1}{p+q}\left(\bigotimes_{\substack{i, j=1 \\
i \neq j}}^{n}\left(\left(\left(p h_{i}^{w_{i}} \oplus q h_{j}^{w_{j}}\right) \otimes\left(p h_{j}^{w_{j}} \oplus q h_{i}^{w_{i}}\right)\right)^{C_{m_{i, j, j<j}}^{w}\left(\substack{-i, j, i<j \\
v_{w}}\right.}\right)\right)^{\frac{2}{n(n-1)}}\right) \tag{1.142}
\end{align*}
$$

where $p, q>0$, then $H F C G B_{w}^{p, q}$ is called a weighted hesitant fuzzy geometric Bonferroni mean (WHFCGBM), and

$$
\begin{align*}
& =\bigcup_{\varepsilon_{i, j}^{w} \in \tau_{i, j, i<j}^{w}}\left\{1-\left(1-\left(\prod_{\substack{i, j=1 \\
i<j}}^{n}\left(\left(\varepsilon_{i, j}^{w}\right)^{c_{m_{i, j, j<j}}^{w}\binom{-i, j, i<j}{v_{w}}}\right)^{\frac{2}{n(n-1)}}\right)\right)^{\frac{1}{p+q}}\right\} \tag{1.143}
\end{align*}
$$

Example 1.21 (Zhu et al. 2012b). Assume that the attributes have correlations with each other, and

$$
\begin{gathered}
\dot{m}_{1,2}(\phi)=\dot{m}_{2,3}(\phi)=\dot{m}_{1,3}(\phi)=0, \dot{m}_{1,3}\left(\left\{h_{2,3}\right\}\right)=0.4 \\
\dot{m}_{1,2}\left(\left\{h_{2,3}\right\}\right)=0.3 \\
\dot{m}_{1,2}\left(\left\{h_{1,3}\right\}\right)=\dot{m}_{1,3}\left(\left\{h_{1,2}\right\}\right)=0.2 \\
\dot{m}_{2,3}\left(\left\{h_{1,2}\right\}\right)=\dot{m}_{2,3}\left(\left\{h_{1,3}\right\}\right)=0.5 \dot{m}_{1,2} \\
\left(\left\{h_{2,3}, h_{1,3}\right\}\right)=\dot{m}_{1,3}\left(\left\{h_{2,3}, h_{1,2}\right\}\right)=\dot{m}_{2,3}\left(\left\{h_{1,2}, h_{1,3}\right\}\right)=1
\end{gathered}
$$

Utilizing the data in Table 1.7, and let $p=q=1$, we have

$$
s\left(h_{1}\right)=0.8927, s\left(h_{2}\right)=0.8823, s\left(h_{3}\right)=0.9334
$$

and thus $A_{3} \succ A_{1} \succ A_{2}$.

### 1.5 Hesitant Fuzzy Aggregation Operators Based on QuasiArithmetic Means and Induced Idea

Quasi-arithmetic means (Hardy et al. 1934) and the induced idea (Yager and Filev 1999) are hot topics in aggregation, about which a lot of work has been done. Yager and Filev (1999) provided a generalization of the process used for ordering the argument values and introduced a more general type of OWA operator (Yager 1988), which they named the induced ordered weighted averaging operator. Merigó and Casanovas (2011a) presented the uncertain induced quasi-arithmetic OWA operator. It is an extension of the OWA operator that uses the main characteristics of the induced OWA, quasi-arithmetic OWA, and uncertain OWA operators (Xu and Da 2002a). Xu and Xia (2011a) investigated the induced generalized aggregation operators under intuitionistic fuzzy environments, including the induced generalized intuitionistic fuzzy Choquet integral operators and the induced generalized intuitionistic fuzzy Dempster-Shafer operators, etc., and gave their application in MADM. See Xu and Cai (2010a, 2012a) for more details on intuitionistic fuzzy aggregation techniques.

Usually, the weight vectors of the aggregation operators are assumed known, not reflecting the correlation of the aggregated arguments. To obtain the weight vector more objectively, the study of the correlation among the aggregation arguments is necessary. More and more researchers have been paying attention to this issue. Yager (2001) introduced the power average to provide an aggregation operator which allows argument values to support each other in the aggregation process, based on which, Xu and Yager (2010) developed a power geometric operator and its weighted form, developed a power ordered geometric operator and a power ordered weighted geometric operator, and studied some of their properties. Xu and Cai (2012b) applied the power aggregation idea to uncertain environments, and gave some methods for group decision making with interval fuzzy preference relations. Xu (2011) developed a series of power aggregation operators for intuitionistic fuzzy information, then applied them to develop some approaches to intuitionistic fuzzy multi-attribute group decision making. In MADM, to reflect the correlation between attributes, Choquet integral (1953) is another important technique providing a type of operator used to measure the expected utility of an uncertain event, and has been applied in many fields. Yager (2004b) introduced the idea of order induced aggregation to Choquet aggregation operator and defined an induced Choquet ordered averaging operator, which allows the ordering of the arguments to be based upon some other associated variables instead of ordering the arguments based on their values. Tan and Chen (2009) developed an induced Choquet ordered averaging operator and applied it to aggregate fuzzy preference relations in group decision making. Xu (2010a), Tan
and Chen (2010) applied Choquet integral to intuitionistic fuzzy environment and proposed some intuitionistic fuzzy aggregation techniques.

It is noted that the number of values in different HFEs may be different, and the values are usually out of order, let $l_{h}$ be the number of values in $h$. Then, we can arrange them in any order for convenience. We arrange the elements in $h$ in decreasing order, and let $h^{\sigma(i)}\left(i=1,2, \cdots, l_{h}\right)$ be the $i$ th smallest value in $h$.

Xu and Xia (2011b,c) defined a distance measure for HFSs, which is also suitable for HFEs described as follows:

Definition 1.30 (Xu and Xia 2011b,c). For two HFEs $h_{1}$ and $h_{2}$, the distance measure between $h_{1}$ and $h_{2}$, denoted as $d\left(h_{1}, h_{2}\right)$, should satisfy the following properties:
(1) $0 \leq d\left(h_{1}, h_{2}\right) \leq 1$.
(2) $d\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}=h_{2}$.
(3) $d\left(h_{1}, h_{2}\right)=d\left(h_{2}, h_{1}\right)$.

Let $l=\max \left\{l_{h_{1}}, l_{h_{2}}\right\}$. To operate correctly, Xu and Xia (2011b,c) gave the following regulation:

When $l_{h_{1}} \neq l_{h_{2}}$, we can make them equivalent through adding values to the HFEs that has less number of elements. In terms of pessimistic principles, the smallest element can be added while the opposite case will be adopted following optimistic principles. In our work, we adopt the former. Specifically, If $l_{h_{1}}<l_{h_{2}}$, then $h_{1}$ should be extended by adding the minimum value in it until it has the same length with $h_{2}$; If $l_{h_{1}}>l_{h_{2}}$, then $h_{2}$ should be extended by adding the minimum value in it until it has the same length with $h_{1}$.

Moreover, we can give another important property that $d\left(h_{1}, h_{2}\right)$ should satisfy, that is:
(4) For three HFEs $h_{1}, h_{2}$ and $h_{3}$, which have the same length $l$, if $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq h_{3}^{\sigma(i)}, i=1,2, \cdots, l$, then

$$
\begin{equation*}
d\left(h_{1}, h_{2}\right) \leq d\left(h_{1}, h_{3}\right), d\left(h_{2}, h_{3}\right) \leq d\left(h_{1}, h_{3}\right) \tag{1.144}
\end{equation*}
$$

Based on the well-known Hamming distance, the hesitant normalized Hamming distance is defined as follows:

$$
\begin{equation*}
d\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right| \tag{1.145}
\end{equation*}
$$

In fact, we have

$$
\begin{equation*}
d\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right| \leq \frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{3}^{\sigma(i)}\right|=d\left(h_{1}, h_{3}\right) \tag{1.146}
\end{equation*}
$$

and

$$
\begin{equation*}
d\left(h_{2}, h_{3}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{2}^{\sigma(i)}-h_{3}^{\sigma(i)}\right| \leq \frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{3}^{\sigma(i)}\right|=d\left(h_{1}, h_{3}\right) \tag{1.147}
\end{equation*}
$$

Similarly, we can prove the other distance measures defined by Xu and Xia (2011b,c) also satisfy (4).

Definition 1.31 (Sugeno 1974; Wang and Klir 1992; Denneberg 1994). A normalized measure $\dot{m}$ on the set $X$ is a function $\dot{m}: \xi(X) \rightarrow[0,1]$ satisfying the following axioms:
(1) $\dot{m}(\phi)=0, \dot{m}(X)=1$.
(2) $X_{1} \subseteq X_{2}$ implies $\dot{m}\left(X_{1}\right) \leq \dot{m}\left(X_{2}\right)$, for all $X_{1}, X_{2} \subseteq X$.
(3) $\dot{m}\left(X_{1} \bigcup X_{2}\right)=\dot{m}\left(X_{1}\right)+\dot{m}\left(X_{2}\right)+\dot{\eta} \dot{m}\left(X_{1}\right) \dot{m}\left(X_{2}\right)$, for all $X_{1}, X_{2} \subseteq X$ and $X_{1} \cap X_{2}=\phi$, where $\dot{\eta} \in(-1, \infty)$.

Especially, if $\dot{\eta}=0$, then (3) in Definition 1.31 reduces to the axiom of additive measure $\dot{m}\left(X_{1} \cup X_{2}\right)=\dot{m}\left(X_{1}\right)+\dot{m}\left(X_{2}\right)$, which indicates that there is no interaction between $X_{1}$ and $X_{2}$; If $\dot{\eta}>0$, then $\dot{m}\left(X_{1} \cup X_{2}\right)>\dot{m}\left(X_{1}\right)+\dot{m}\left(X_{2}\right)$, which implies that the set $\left\{X_{1}, X_{2}\right\}$ has multiplicative effect; If $\dot{\eta}<0$, then $\dot{m}\left(X_{1} \cup X_{2}\right)<\dot{m}\left(X_{1}\right)+\dot{m}\left(X_{2}\right)$, which implies that the set $\left\{X_{1}, X_{2}\right\}$ has substitutive effect, by the parameter $\dot{\eta}$, the interaction between sets or elements of set can be represented.

Let $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ be a finite set, then $\bigcup_{i=1}^{n} x_{i}=X$. To determine the normalized measure on $X$ avoiding the computational complexity, Sugeno (1974) gave the following equation:

$$
\dot{m}(X)=\dot{m}\left(\bigcup_{i=1}^{n} x_{i}\right)= \begin{cases}\frac{1}{\dot{\eta}}\left(\prod_{i=1}^{n}\left(1+\dot{\eta} \dot{m}\left(x_{i}\right)\right)-1\right), & \dot{\eta} \neq 0  \tag{1.148}\\ \sum_{i=1}^{n} \dot{m}\left(x_{i}\right), & \dot{\eta}=0\end{cases}
$$

and the value of $\dot{\eta}$ can be uniquely determined from $\dot{m}(X)=1$, which can be written as:

$$
\begin{equation*}
\dot{\eta}+1=\prod_{i=1}^{n}\left(1+\dot{\eta} \dot{m}\left(x_{i}\right)\right) \tag{1.149}
\end{equation*}
$$

Especially, for every subset $\bar{X} \subseteq X$, we have

$$
\dot{m}(\bar{X})= \begin{cases}\frac{1}{\dot{\eta}}\left(\prod_{x_{i} \in \bar{X}}\left(1+\dot{\eta} \dot{m}\left(x_{i}\right)\right)-1\right), & \dot{\eta} \neq 0  \tag{1.150}\\ \sum_{x_{i} \in \bar{X}} \dot{m}\left(x_{i}\right), & \dot{\eta}=0\end{cases}
$$

The aggregation (Fodor et al. 1995) based on the quasi-arithmetic means (Hardy et al. 1934) generalizes a wide range of aggregation operators, and has been extended to aggregate many kinds of fuzzy information (Wang and Hao 2006; Merigó and Gil-Lafuente 2009; Xu and Xia 2011a; Merigó and Casanovas 2011a). In this section, we apply the quasi-arithmetic means to aggregate hesitant fuzzy information:

Definition 1.32 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ the weight vector of $h_{i}(i=1,2, \cdots, n)$, such that $\sum_{i=1}^{n} w_{i}=1$ and $w_{i} \geq 0, i=1,2, \cdots, n$. Let QHFWA: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\phi^{-1}\left(\sum_{i=1}^{n} w_{i} \phi\left(\gamma_{i}\right)\right)\right\} \tag{1.151}
\end{equation*}
$$

then QHFWA is called a quasi hesitant fuzzy weighted agregation (QHFWA) operator, where $g(\gamma)$ is a strictly continuous monotonic function.

It is noted that, when $g(\gamma)$ is given different functions, we can get different aggregation operators, such as:
(1) If $\phi(\gamma)=\gamma^{\lambda}$, then

$$
\begin{equation*}
\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(\sum_{i=1}^{n} w_{i} \gamma_{i}^{\lambda}\right)^{\frac{1}{\lambda}}\right\} \tag{1.152}
\end{equation*}
$$

(2) If $\phi(\gamma)=1-(1-\gamma)^{\lambda}$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\left(\sum_{i=1}^{n} w_{i}\left(1-\gamma_{i}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right\}$
(3) If $\phi(\gamma)=\sin \left(\frac{\pi}{2} \gamma\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\pi}{2} \arcsin \left(\sum_{i=1}^{n} w_{i} \sin \frac{\pi \gamma_{i}}{2}\right)\right\}$
(4) If $\phi(\gamma)=1-\sin \left(\frac{\pi}{2}(1-\gamma)\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\frac{\pi}{2} \arcsin \left(\sum_{i=1}^{n} w_{i} \sin \frac{\pi\left(1-\gamma_{i}\right)}{2}\right)\right\}$
(5) If $\phi(\gamma)=\cos \left(\frac{\pi}{2} \gamma\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\pi}{2} \arccos \left(\sum_{i=1}^{n} w_{i} \cos \frac{\pi \gamma_{i}}{2}\right)\right\}$
(6) If $\phi(\gamma)=1-\cos \left(\frac{\pi}{2}(1-\gamma)\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\frac{\pi \gamma}{2} \arccos \left(\sum_{i=1}^{n} w_{i} \cos \frac{\pi\left(1-\gamma_{i}\right)}{2}\right)\right\}$
(7) If $\phi(\gamma)=\tan \left(\frac{\pi}{2} \gamma\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\pi}{2} \arctan \left(\sum_{i=1}^{n} w_{i} \tan \frac{\pi \gamma_{i}}{2}\right)\right\}$
(8) If $\phi(\gamma)=1-\tan \left(\frac{\pi}{2}(1-\gamma)\right)$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\frac{\pi}{2} \arctan \left(\sum_{i=1}^{n} w_{i} \tan \frac{\pi\left(1-\gamma_{i}\right)}{2}\right)\right\}$
(9) If $\phi(\gamma)=\lambda^{\gamma}, b, \lambda>0, \lambda \neq 1$, then

$$
\begin{equation*}
\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\log _{b}\left(\sum_{i=1}^{n} w_{i} \lambda^{\gamma_{i}}\right)\right\} \tag{1.160}
\end{equation*}
$$

(10) If $\phi(\gamma)=1-b^{1-\gamma}, b, \lambda>0, \lambda \neq 1$, then

$$
\begin{equation*}
\operatorname{QHFW}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\log _{b}\left(\sum_{i=1}^{n} w_{i} \lambda^{1-\gamma_{i}}\right)\right\} \tag{1.161}
\end{equation*}
$$

(11) If $\phi(\gamma)=\frac{\gamma^{\lambda}}{\gamma^{\lambda}+(1-\gamma)^{\lambda}}$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\left(\sum_{i=1}^{n} \frac{w_{i} \gamma_{i}^{\lambda}}{\gamma_{i}^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}{\left(\sum_{i=1}^{n} \frac{w_{i} \gamma_{i}^{\lambda}}{\gamma_{i}^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}+\left(1-\sum_{i=1}^{n} \frac{w_{i} \gamma_{i}^{\lambda}}{\gamma_{i}^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.162}
\end{equation*}
$$

(12) If $\phi(\gamma)=\frac{(1+\gamma)^{\lambda}-(1-\gamma)^{\lambda}}{(1+\gamma)^{\lambda}+(1-\gamma)^{\lambda}}, \lambda>0$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{align*}
& =\gamma_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}^{U}\left\{\frac{\left(1+\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}-\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}-\left(1-\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}-\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}{\left(1+\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}-\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}+\left(1-\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}-\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right\} \\
& =\gamma_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}^{\bigcup}\left\{\frac{\left(\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}-\left(\sum_{i=1}^{n} w_{i} \frac{\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}{\left.\left(\sum_{i=1}^{n} w_{i} \frac{\left(1+\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}+\left(\sum_{i=1}^{n} w_{i} \frac{\left(1-\gamma_{i}\right)^{\lambda}}{\left(1+\gamma_{i}\right)^{\lambda}+\left(1-\gamma_{i}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}\right\}}\right. \tag{1.163}
\end{align*}
$$

(13) If $\boldsymbol{\phi}(\gamma)=\frac{2 \gamma^{\lambda}}{(2-\gamma)^{\lambda}+\gamma^{\lambda}}, \lambda>0$, then
$\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{align*}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{2\left(\sum_{i=1}^{n} w_{i} \frac{2 \gamma_{i}^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}}{\left(2-\sum_{i=1}^{n} w_{i} \frac{2 \gamma_{i}^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}+\left(\sum_{i=1}^{n} w_{i} \frac{2 \gamma_{i}^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}}\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}^{U}\left\{\frac{2\left(\sum_{i=1}^{n} w_{i} \frac{\gamma_{i}^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}}{\left(\sum_{i=1}^{n} w_{i} \frac{\left(2-\gamma_{i}\right)^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}+\left(\sum_{i=1}^{n} w_{i} \frac{\gamma_{i}^{\lambda}}{\left(2-\gamma_{i}\right)^{\lambda}+\gamma_{i}^{\lambda}}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.164}
\end{align*}
$$

It is pointed out that in (1), (2), (9)-(13), if $\lambda=1$, then

$$
\begin{equation*}
\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\sum_{i=1}^{n} w_{i} \gamma_{i}\right\} \tag{1.165}
\end{equation*}
$$

In fact, based on the ordered modular averages (OMAs) proposed by Mesiar and Mesiarová-Zemánková (2011), we can further generalize the QHFWA operator as follows:

Definition 1.33 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ the weight vector of $h_{i}(i=1,2, \cdots, n)$, such that $\sum_{i=1}^{n} w_{i}=1$ and $w_{i} \geq 0, i=1,2, \cdots, n$. Let HFMWA: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\sum_{i=1}^{n} w_{i} \phi_{i}\left(\gamma_{i}\right)\right\} \tag{1.166}
\end{equation*}
$$

then HFMWA is called a hesitant fuzzy modular weighted averaging (QHFWA) operator, where $\phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, which can be replaced by the functions discussed in Eqs.(1.152)-(1.164).

Definition 1.34 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ the weight vector of $h_{i}(i=1,2, \cdots, n)$, such that $\sum_{i=1}^{n} w_{i}=1$ and $w_{i} \geq 0, i=1,2, \cdots, n$. Let HFMWG: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\phi_{i}\left(\gamma_{i}\right)\right)^{w_{i}}\right\} \tag{1.167}
\end{equation*}
$$

then HFMWG is called the hesitant fuzzy modular weighted geometric (HFMWG) operator, where $\phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, which can be replaced by the functions discussed in Eqs.(1.152)-(1.164).

Especially, if $\phi_{1}(\gamma)=\phi_{2}(\gamma)=\cdots=\phi_{n}(\gamma)=\phi(\gamma)$, then the HFMWA and HFMWG operators reduce to the following:

$$
\begin{align*}
& \operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\sum_{i=1}^{n} w_{i} \phi\left(\gamma_{i}\right)\right\}  \tag{1.168}\\
& \operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\phi\left(\gamma_{i}\right)\right)^{w_{i}}\right\} \tag{1.169}
\end{align*}
$$

If $\phi_{1}(\gamma)=\phi_{2}(\gamma)=\cdots=\phi_{n}(\gamma)=\gamma$, then the HFMWA and HFMWG operators reduce to the following:

$$
\begin{align*}
& \operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\sum_{i=1}^{n} w_{i} \gamma_{i}\right\}  \tag{1.170}\\
& \operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n} \gamma_{i}^{w_{i}}\right\} \tag{1.171}
\end{align*}
$$

In the above defined aggregation operators (1.151)-(1.171), the weight vectors are assumed known, however, in many practical problems, they are unknown and the aggregated arguments have connections among them, and the weight vectors should better reflect this issue.

Motivated by the power average (PA) operator (Yager 2001), we give one method to determine the weight vector of the proposed aggregation operators:

Let

$$
\begin{equation*}
w_{i}=\frac{\left(1+T\left(h_{i}\right)\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right)}, i=1,2, \cdots, n \tag{1.172}
\end{equation*}
$$

where

$$
\begin{equation*}
T\left(h_{i}\right)=\sum_{\substack{j=1 \\ j \neq i}}^{n} S u p\left(h_{i}, h_{j}\right), i=1,2, \cdots, n \tag{1.173}
\end{equation*}
$$

and $\operatorname{Sup}\left(h_{i}, h_{j}\right)$ is the support for $h_{i}$ from $h_{j}$, with the conditions:
(1) $\operatorname{Sup}\left(h_{i}, h_{j}\right) \in[0,1]$.
(2) $\operatorname{Sup}\left(h_{i}, h_{j}\right)=\operatorname{Sup}\left(h_{j}, h_{i}\right)$.
(3) $\operatorname{Sup}\left(h_{i}, h_{j}\right) \geq \operatorname{Sup}\left(h_{k_{1}}, h_{k_{2}}\right)$, if $d\left(h_{i}, h_{j}\right)<d\left(h_{k_{1}}, h_{k_{2}}\right)$, where $d$ is a distance measure.

Let $\operatorname{Sup}\left(h_{i}, h_{j}\right)=a$, for all $i \neq j$, then

$$
\begin{equation*}
w_{i}=\frac{\left(1+T\left(h_{i}\right)\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right)}=\frac{1}{n}, i=1,2, \cdots, n \tag{1.174}
\end{equation*}
$$

which indicates that when all the supports are the same, then the weights of arguments are also the same.

If we combine the QHFWA, HFMWA and HFMWG operators, respectively, then we have

$$
\begin{align*}
& \operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\phi^{-1}\left(\frac{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right) \phi\left(\gamma_{i}\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right)}\right]\right\}  \tag{1.175}\\
& \operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right) \phi_{i}\left(\gamma_{i}\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{i}\right)\right)}\right\}  \tag{1.176}\\
& \operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\phi_{i}\left(\gamma_{i}\right)\right)^{\frac{1+1}{\sum_{i=1}^{n}\left(1+\tau\left(h_{i}\right)\right.}}\right) \tag{1.177}
\end{align*}
$$

Based on Choquet integral, we can let $w_{i}=\dot{m}\left(X_{i}\right)-\dot{m}\left(X_{i-1}\right)$, where $X_{i}=\left\{x_{1}, x_{2}, \cdots, x_{i}\right\}$ when $i \geq 1$ and $X_{0}=\varnothing$, and in this case, we have

$$
\begin{equation*}
\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{i} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\phi^{-1}\left(\sum_{i=1}^{n}\left(\dot{m}\left(X_{i}\right)-\dot{m}\left(X_{i-1}\right)\right) \phi\left(\gamma_{i}\right)\right)\right\} \tag{1.178}
\end{equation*}
$$

$$
\begin{equation*}
\operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\sum_{i=1}^{n}\left(\dot{m}\left(X_{i}\right)-\dot{m}\left(X_{i-1}\right)\right) \phi_{i}\left(\gamma_{i}\right)\right\} \tag{1.179}
\end{equation*}
$$

$\operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\phi_{i}\left(\gamma_{i}\right)\right)^{\dot{m}\left(X_{i}\right)-\dot{m}\left(X_{i-1}\right)}\right\}$
We can find that although both of these two weight-determined methods reflect the connections of the arguments, the former is more objective, because that it is based on the support degree between each two pair of arguments, while the latter is only based on the given normalized measures. Thus, the former can be used to
determine the weight vector of the DMs in group decision making by calculating the distance or similarity between each other, while the latter can be used to determine the weight vector of attributes by measuring the correlations among them.

If arguments are ordered first before being aggregated, we can get another series of popular aggregation operators, the OWA (Yager 1988) operators. In this section, we extend the OWA operators to aggregate hesitant fuzzy information.

Definition 1.35 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $h_{\sigma(i)}$ the $i$ th largest of them. Let QHFOWA: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{QHFOWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\sum_{i=1}^{n} \omega_{i} \phi\left(\gamma_{\sigma(i)}\right)\right)\right\} \tag{1.181}
\end{equation*}
$$

then QHFOWA is called a quasi hesitant fuzzy ordered weighted aggregation (QHFOWA) operator, where $\phi(\gamma)$ is a strictly continuous monotonic function, $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$, and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Similarly, Liao and Xu (2013c) gave the QHFOWG operator as follows:
Definition 1.36 (Liao and Xu 2013 c ). Let $h_{j}(j=1,2, \cdots, n)$ be a collection of HFEs and $h_{\sigma(j)}$ the $j$ th largest of them. Let QHFOWG: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{QHFOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\prod_{j=1}^{n} \phi^{\omega_{j}}\left(\gamma_{\sigma(j)}\right)\right)\right\} \tag{1.182}
\end{equation*}
$$

then QHFOWG is called a quasi hesitant fuzzy ordered weighted geometric (QHFOWG) operator, where $\phi(\gamma)$ is a strictly continuous monotonic function, $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$, and $\omega_{j} \geq 0, j=1,2, \ldots, n$.

Definition 1.37 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $h_{\sigma(i)}$ the $i$ th largest of them. Let HFMOWA: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{HFMOWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{(2)} \in h_{\sigma(2)}, \cdots, \gamma_{(x)} \in h_{\sigma(n)}}\left\{\sum_{i=1}^{n} \omega_{i} \phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right\} \tag{1.183}
\end{equation*}
$$

then HFMOWA is called a hesitant fuzzy modular ordered weighted averaging (QHFOWA) operator, where $\phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, which can be replaced by the functions discussed in Eqs.(1.152)(1.164), $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$, and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Definition 1.38 (Xia et al. 2013a). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $h_{\sigma(i)}$ the $i$ th largest of them. Let HFMOWG: $\Theta^{n} \rightarrow \Theta$, if

$$
\begin{equation*}
\operatorname{HFMOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\prod_{i=1}^{n}\left(\phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right)^{\omega_{i}}\right\} \tag{1.184}
\end{equation*}
$$

then HFMOWG is called a hesitant fuzzy modular ordered weighted geometric (HFMWG) operator, where $\phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, which can be replaced by the functions discussed in Eqs.(1.152)(1.164), $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$, and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Now we can give two methods to determine the weight vectors associated with the QHFOWA, QHFOWG, HFMOWA and HFMOWG operators:

Considering the support degrees between the aggregated arguments, $\omega_{i}(i=$ $1,2, \ldots, n)$ can be a collection of weights such that

$$
\begin{equation*}
\omega_{i}=g\left(\frac{G_{i}}{T V}\right)-g\left(\frac{G_{i-1}}{T V}\right), G_{i}=\sum_{j=1}^{i} V_{\sigma(j)}, T V=\sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)}=1+T\left(h_{\sigma(i)}\right) \tag{1.185}
\end{equation*}
$$

and $T\left(h_{\sigma(i)}\right)$ denotes the support of the $i$ th largest HFE $h_{\sigma(i)}$ by all the other HFEs, i.e.,

$$
\begin{equation*}
T\left(h_{\sigma(i)}\right)=\sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{Sup}\left(h_{\sigma(i)}, h_{\sigma(j)}\right) \tag{1.186}
\end{equation*}
$$

where $\operatorname{Sup}\left(h_{\sigma(i)}, h_{\sigma(j)}\right)$ indicates the support of $i$ th largest HFE $h_{\sigma(i)}$ for the $j$ th largest HFE $h_{\sigma(j)}$, and $g:[0,1] \rightarrow[0,1]$ is a BUM function, having the properties:
(1) $g(0)=0$.
(2) $g(1)=1$.
(3) $g(x) \geq g(y)$, if $x>y$.

Especially, if $g(x)=x$, then

QHFOWA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\frac{\sum_{i=1}^{n}\left(1+T\left(h_{\sigma(i)}\right)\right) \phi\left(\gamma_{\sigma(i)}\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{\sigma(i)}\right)\right)}\right)\right\}
$$

$$
\begin{equation*}
=\operatorname{QHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.187}
\end{equation*}
$$

$\operatorname{QHFOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{align*}
& =\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\prod_{j=1}^{n}\left(\phi\left(\gamma_{\sigma(j)}\right)\right)^{\frac{1+1\left(h_{\sigma(j)}\right)}{\sum_{i=1}^{n}\left(1+\tau\left(h_{\sigma(i))}\right)\right.}}\right)\right\} \\
& =\operatorname{QHFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.188}
\end{align*}
$$

HFMOWA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{align*}
& =\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\frac{\sum_{i=1}^{n}\left(1+T\left(h_{\sigma(i)}\right)\right) \phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{\sigma(i)}\right)\right)}\right. \\
& =\operatorname{HFMWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.189}
\end{align*}
$$

$\operatorname{HFMOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{align*}
& =\operatorname{HFMWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.190}
\end{align*}
$$

Utilizing Choquet integral, we have $\omega_{i}=\dot{m}\left(X_{\sigma(i)}\right)-\dot{m}\left(X_{\sigma(i-1)}\right)$, where $\sigma$ : $\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\}$ is a permutation such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \cdots \geq h_{\sigma(n)}$, $X_{\sigma(i)}=\left\{x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(i)}\right\}$, when $i \geq 1$ and $X_{\sigma(0)}=\varnothing$. In such cases, we have

QHFOWA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\sum_{i=1}^{n}\left(\left(\dot{m}\left(X_{\sigma(i)}\right)-\dot{m}\left(X_{\sigma(i-1)}\right)\right) \phi\left(\gamma_{\sigma(i)}\right)\right)\right)\right\} \tag{1.191}
\end{equation*}
$$

QHFOWG $\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\prod_{j=1}^{n}\left(\phi\left(\gamma_{\sigma(j)}\right)\right)^{\dot{m}\left(X_{\sigma(i)}\right)-\dot{m}\left(X_{\sigma(i-1)}\right)}\right)\right\} \tag{1.192}
\end{equation*}
$$

HFMOWA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\sum_{i=1}^{n}\left(\left(\dot{m}\left(X_{\sigma(i)}\right)-\dot{m}\left(X_{\sigma(i-1)}\right)\right) \phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right)\right\} \tag{1.193}
\end{equation*}
$$

$\operatorname{HFMOWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\prod_{i=1}^{n}\left(\phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right)^{\dot{m}\left(X_{\sigma(i)}\right)-\dot{m}\left(X_{\sigma(i-1)}\right)}\right\} \tag{1.194}
\end{equation*}
$$

If the reordering step in aggregation operators is not defined by the aggregation arguments, but by order-inducing variables, i.e. the ordered positions of the arguments depend upon the values of the order-inducing variables, then we can get some more general aggregation operators for hesitant fuzzy information.

Definition 1.39 (Xia et al. 2013a). Let $\left\langle u_{i}, h_{i}\right\rangle(i=1,2, \cdots, n)$ be a collection of 2-tuple aggregation arguments, in which $u_{i}$ is referred to as the order-inducing variable and $h_{i}$ as the argument variable represented by HFEs. Let IQHFOWA: $\Theta^{n} \rightarrow \Theta$, if
$\operatorname{IQHFOWA}\left(<u_{1}, h_{1}>,<u_{2}, h_{2}>, \cdots,<u_{n}, h_{n}>\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\sum_{i=1}^{n} \omega_{i} \phi\left(\gamma_{\sigma(i)}\right)\right)\right\} \tag{1.195}
\end{equation*}
$$

then IQHFOWA is called an induced quasi hesitant fuzzy ordered weighted aggregation (IQHFOWA) operator, where $<u_{\sigma(i)}, h_{\sigma(i)}>$ is the 2-tuple with $u_{\sigma(i)}$ the $i$ th largest value in the set $\left\{u_{1}, u_{2}, \cdots, u_{n}\right\}, \phi(\gamma)$ is a strictly continuous monotonic function and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$ and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Definition 1.40. Let $<u_{i}, h_{i}>(i=1,2, \cdots, n)$ be a collection of 2-tuple aggregation arguments, in which $u_{i}$ is referred to as the order-inducing variable
and $h_{i}$ as the argument variable represented by HFEs. Let IQHFOWG: $\Theta^{n} \rightarrow \Theta$, if
$\operatorname{IQHFOWG}\left(<u_{1}, h_{1}>,<u_{2}, h_{2}>, \cdots,<u_{n}, h_{n}>\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\phi^{-1}\left(\prod_{i=1}^{n} \phi^{\omega_{i}}\left(\gamma_{\sigma(i)}\right)\right)\right\} \tag{1.196}
\end{equation*}
$$

then IQHFOWG is called an induced quasi hesitant fuzzy ordered weighted geometric (IQHFOWG) operator, where $<u_{\sigma(i)}, h_{\sigma(i)}>$ is the 2-tuple with $u_{\sigma(i)}$ the $i$ th largest value in the set $\left\{u_{1}, u_{2}, \cdots, u_{n}\right\}, \phi(\gamma)$ is a strictly continuous monotonic function and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$ and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Definition 1.41 (Xia et al. 2013a). Let $\left\langle u_{i}, h_{i}\right\rangle(i=1,2, \cdots, n)$ be a collection of 2-tuple aggregation arguments, in which $u_{i}$ is referred to as the order-inducing variable and $h_{i}$ as the argument variable represented by HFEs. Let IHFMOWA: $\Theta^{n} \rightarrow \Theta$, if

IHFMOWA $\left(<u_{1}, h_{1}>,<u_{2}, h_{2}>, \cdots,<u_{n}, h_{n}>\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\sum_{i=1}^{n} \omega_{i} \phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right\} \tag{1.197}
\end{equation*}
$$

then IHFMOWA is called an induced hesitant fuzzy modular ordered weighted averaging (IHFMOWA) operator, where $<u_{\sigma(i)}, h_{\sigma(i)}>$ is the 2-tuple with $u_{\sigma(i)}$ the $i$ th largest value in the set $\left\{u_{1}, u_{2}, \cdots, u_{n}\right\}, \phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, which can be replaced by the functions discussed in Eqs.(1.152)-(1.164) and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$ and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Definition 1.42 (Xia et al. 2013a). Let $\left\langle u_{i}, h_{i}>(i=1,2, \cdots, n)\right.$ be a collection of 2-tuple aggregation arguments, in which $u_{i}$ is referred to as the order-inducing variable and $h_{i}$ as the argument variable represented by HFEs. Let IHFMOWG: $\Theta^{n} \rightarrow \Theta$, if
$\operatorname{IHFMOWG}\left(<u_{1}, h_{1}>,<u_{2}, h_{2}>, \cdots,<u_{n}, h_{n}>\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\prod_{i=1}^{n}\left(\phi_{\sigma(i)}\left(\gamma_{\sigma(i)}\right)\right)^{v_{i}}\right\} \tag{1.198}
\end{equation*}
$$

then IHFMOWG is called an induced hesitant fuzzy modular ordered weighted geometric (IHFMWG) operator, where $<u_{\sigma(i)}, h_{\sigma(i)}>$ is the 2-tuple with $u_{\sigma(i)}$ the $i$ th largest value in the set $\left\{u_{1}, u_{2}, \cdots, u_{n}\right\}, \phi_{i}(i=1,2, \cdots, n)$ are strictly continuous monotonic functions, and $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ is the associated weight vector with $\sum_{i=1}^{n} \omega_{i}=1$ and $\omega_{i} \geq 0, i=1,2, \ldots, n$.

Then the weight vector $\omega_{i}(i=1,2, \ldots, n)$ can be defined as:

$$
\begin{equation*}
\omega_{i}=g\left(\frac{\dot{G}_{i}}{T V}\right)-g\left(\frac{\dot{G}_{i-1}}{T V}\right), \dot{G}_{i}=\sum_{j=1}^{i} V_{\sigma(j)}, T V=\sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)}=1+T\left(h_{\sigma(i)}\right) \tag{1.199}
\end{equation*}
$$

and $T\left(h_{\sigma(i)}\right)$ denotes the support of the $i$ th largest HFE $h_{\sigma(i)}$ by all the other HFEs, i.e.,

$$
\begin{equation*}
T\left(h_{\sigma(i)}\right)=\sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{Sup}\left(h_{\sigma(i)}, h_{\sigma(j)}\right) \tag{1.200}
\end{equation*}
$$

where $\sigma: \quad\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\} \quad$ is $\quad$ a permutation such that $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}, \quad \operatorname{Sup}\left(h_{\sigma(i)}, h_{\sigma(j)}\right)$ indicates the support of $j$ th largest HFE $h_{\sigma(i)}$ for the $i$ th largest HFE $h_{\sigma(j)}$, and $g:[0,1] \rightarrow[0,1]$ is a BUM function as defined previously.

The weight vector $\omega_{i}(i=1,2, \ldots, n)$ can also be defined as $\omega_{i}=\dot{m}\left(X_{\sigma(i)}\right)-$ $\dot{m}\left(X_{\sigma(i-1)}\right)$, where $\sigma:\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\}$ is a permutation such that $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}, X_{\sigma(i)}=\left\{x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(i)}\right\}, i \geq 1$ and $X_{\sigma(0)}=\varnothing$.

Consider a multi-attribute group decision making problem under uncertainty. Suppose that there are $n$ alternatives $A_{i}(i=1,2, \cdots, n)$ and $m$ attributes $x_{j}(j=1,2, \cdots, m)$, and let $D=\left\{D_{1}, D_{2}, \ldots, D_{p_{0}}\right\}$ be the set of DMs. Suppose that the DM $D_{k}$ provides all the possible evaluated values under the attribute $x_{j}$ for the alternative $A_{i}$ denoted by a HFE $h_{i j}^{(k)}$ and constructs the decision matrix $H_{k}=\left(h_{i j}^{(k)}\right)_{n \times m}$. Then, based on the developed aggregation operators, we give a method for multi-attribute group decision making with hesitant fuzzy information, which involves the following steps:

Step 1. Let $\operatorname{Sup}\left(h_{i j}^{(k)}, h_{i j}^{(l)}\right)=1-d\left(h_{i j}^{(k)}, h_{i j}^{(l)}\right)$, and calculate the support $T\left(h_{i j}^{(k)}\right)$ of the evaluated value $h_{i j}^{(k)}$ by the other evaluated values $h_{i j}^{(l)}$ $\left(l=1,2, \ldots, p_{0}, l \neq k\right)$ :

$$
\begin{equation*}
T\left(h_{i j}^{(k)}\right)=\sum_{\substack{l=1 \\ l \neq k}}^{p_{0}} \operatorname{Sup}\left(h_{i j}^{(k)}, h_{i j}^{(l)}\right) \tag{1.201}
\end{equation*}
$$

and calculate the weights $w_{i j}^{(k)}\left(k=1,2, \ldots, p_{0}\right)$ associated with the evaluated values $h_{i j}^{(k)}\left(k=1,2, \ldots, p_{0}\right)$ :

$$
\begin{equation*}
w_{i j}^{(k)}=\frac{1+T\left(h_{i j}^{(k)}\right)}{\sum_{k=1}^{p_{0}}\left(1+T\left(h_{i j}^{(k)}\right)\right)}, k=1,2, \ldots, p_{0} \tag{1.202}
\end{equation*}
$$

where $w_{i j}^{(k)} \geq 0, k=1,2, \ldots, p_{0}$, and $\sum_{k=1}^{p_{0}} w_{i j}^{(k)}=1$.
Step 2. Based on Choquet integral, we calculate the correlations between the attributes using the method given previously (many methods have been developed upon this issue, i.e., Tan and Chen (2009) used the normalized measure given by Sugeno (1974) to determine them (See Section 1.5); Büyüközkan et al. (2009, 2010) used the 2-additive measure (Grabisch 1997; Büyüközkan et al. 2003) to determine them).

Step 3. Aggregate all the individual hesitant fuzzy decision matrix $H_{k}=$ $\left(h_{i j}^{(k)}\right)_{n \times m}\left(k=1,2, \ldots, p_{0}\right)$ into the collective hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{n \times m}$, where

$$
\begin{gather*}
h_{i j}=\bigcup_{\gamma_{i j}^{(1)} \in h_{i j}^{(1)}, \gamma_{i j}^{(2)} \in h_{i j}^{(2)}, \cdots, \gamma_{i j}^{(p)} \in h_{i j}^{(p)}}\left\{\phi^{-1}\left(\sum_{k=1}^{p_{0}} w_{i j}^{(k)} \phi\left(\gamma_{i j}^{(k)}\right)\right)\right\}, \\
i=1,2, \ldots, n ; j=1,2, \ldots, m  \tag{1.203}\\
h_{i j}=\bigcup_{\gamma_{i j}^{(1)} \in h_{i j}^{(1)}, \gamma_{i j}^{(2)} \in h_{i j}^{(2)}, \cdots, \gamma_{i j}^{(p)} \in h_{i j}^{(p)}}\left\{\phi ^ { - 1 } \left(\prod_{k=1}^{p_{0}}\left(\phi\left(\gamma_{i j}^{(k)}\right)\right)^{\left.\left.w_{i j}^{(k)}\right)\right\},}\right.\right. \\
i=1,2, \ldots, n ; j=1,2, \ldots, m  \tag{1.204}\\
h_{i j}=\bigcup_{\left.\gamma_{i j}^{(i)} \in h_{i j}^{(1)}, \gamma_{i j}^{(2)} \in h_{i j}^{(2)}, \cdots, \gamma_{i j}^{(p)^{(p)} \in h_{i j}^{(p)}}\right)}\left\{\sum_{k=1}^{p_{0}} w_{i j}^{(k)} \phi_{k}\left(\gamma_{i j}^{(k)}\right)\right\}, \\
i=1,2, \ldots, n ; j=1,2, \ldots, m \tag{1.205}
\end{gather*}
$$

or

$$
\begin{align*}
h_{i j}= & \bigcup_{\gamma_{i j}^{(1)} \in h_{i j}^{(1)}, \gamma_{i j}^{(2)} \in h_{i j}^{(2)}, \cdots, \gamma_{i j}^{\left(p_{0}\right)} \in h_{i j}^{\left(p_{0}\right)}}
\end{align*}\left\{\prod_{k=1}^{p_{0}}\left(\phi_{k}\left(\gamma_{i j}^{(k)}\right)\right)^{\left.w_{i j}^{(k)}\right\}}\right\},
$$

Step 4. Get the expected results $h_{i}$ for the alternatives $A_{i}(i=1,2, \ldots, n)$ :

$$
\begin{gather*}
h_{i}=\bigcup_{\gamma_{i 1} \in h_{i 1}, \gamma_{i 2} \in h_{i 2}, \cdots, \gamma_{i m} \in h_{i m}}^{\bigcup}\left\{\phi^{-1}\left(\sum_{j=1}^{m}\left(\dot{m}\left(X_{j}\right)-\dot{m}\left(X_{j-1}\right)\right) \phi\left(\gamma_{i j}\right)\right)\right\}, i=1,2, \ldots, n  \tag{1.207}\\
h_{i}=\bigcup_{\gamma_{i 1} \in h_{i 1}, \gamma_{i 2} \in h_{i 2}, \cdots, \gamma_{i m} \in h_{i m}}^{\bigcup}\left\{\phi^{-1}\left(\prod_{j=1}^{m}\left(\phi\left(\gamma_{i j}\right)\right)^{\dot{m}\left(X_{j}\right)-\dot{m}\left(X_{j-1}\right)}\right)\right\}, i=1,2, \ldots, n \tag{1.208}
\end{gather*}
$$

$$
\begin{equation*}
h_{i}=\bigcup_{\gamma_{i 1} \in h_{i 1}, \gamma_{i 2} \in h_{i 2}, \cdots, \gamma_{i m} \in h_{i m}}\left\{\sum_{j=1}^{m}\left(\dot{m}\left(X_{j}\right)-\dot{m}\left(X_{j-1}\right)\right) \phi_{j}\left(\gamma_{i j}\right)\right\}, i=1,2, \ldots, n \tag{1.209}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i}=\bigcup_{\gamma_{i 1} \in h_{i 1}, \gamma_{i 2} \in h_{i 2}, \cdots, \gamma_{i m} \in h_{i m}}\left\{\prod_{k=1}^{p_{0}}\left(\phi_{j}\left(\gamma_{i j}\right)\right)^{\dot{m}\left(X_{j}\right)-\dot{m}\left(X_{j-1}\right)}\right\}, i=1,2, \ldots, n \tag{1.210}
\end{equation*}
$$

and get the overall preference value $h_{i}$ corresponding to the alternative $A_{i}$.

Step 5. Calculate the scores of $h_{i}$, and rank the alternatives according to $s\left(h_{i}\right)$ $(i=1,2, \ldots, n)$.

To illustrate the proposed method, an example (adapted from Chen (2011)) is given as follows:

Example 1.22 (Xia et al. 2013a). The following practical example involves a supplier selection problem in a supply chain. The authorized DMs in a small enterprise attempt to reduce the supply chain risk and uncertainty to improve customer service, inventory levels, and cycle times, which results in increased competitiveness and profitability. The DMs consider various criteria involving: (1) $x_{1}$ : Performance (e.g., delivery, quality, price); (2) $x_{2}$ : Technology (e.g., manufacturing capability, design capability, ability to cope with technology changes); (3) $x_{3}$ : Organizational culture and strategy (e.g., feeling of trust, internal and external integration of suppliers, compatibility across levels and functions of three suppliers: $A_{i}(i=1,2,3)$. There are three DMs $D_{i}(i=1,2,3)$, are authorized to evaluate these five suppliers. Suppose the DMs $D_{i}(i=1,2,3)$ provide all the possible evaluated values under the attributes $x_{j}(j=1,2,3)$ for the alternatives $A_{i}(i=1,2,3)$ denoted by the HFEs $h_{i j}^{(k)}$ ( $i=1,2,3 ; j=1,2,3 ; k=1,2,3$ ) and construct the decision matrices $H_{k}=$ $\left(h_{i j}^{(k)}\right)_{5 \times 3}(k=1,2,3)$ (see Tables 1.8-1.10 (Xia et al. 2013a)).

Table 1.8. The hesitant fuzzy decision matrix $H_{1}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.6\}$ | $\{0.7\}$ | $\{0.4,0.5\}$ |
| $A_{2}$ | $\{0.6,0.8\}$ | $\{0.5,0.9\}$ | $\{0.7\}$ |
| $A_{3}$ | $\{0.4,0.5\}$ | $\{0.3\}$ | $\{0.6\}$ |

Table 1.9. The hesitant fuzzy decision matrix $\mathrm{H}_{2}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4\}$ | $\{0.3,0.5\}$ | $\{0.4\}$ |
| $A_{2}$ | $\{0.8\}$ | $\{0.7\}$ | $\{0.6,0.7\}$ |
| $A_{3}$ | $\{0.4\}$ | $\{0.3,0.6\}$ | $\{0.5,0.7\}$ |

Table 1.10. The hesitant fuzzy decision matrix $\mathrm{H}_{3}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5\}$ | $\{0.3,0.4\}$ | $\{0.6\}$ |
| $A_{2}$ | $\{0.7,0.9\}$ | $\{0.8\}$ | $\{0.5,0.6\}$ |
| $A_{3}$ | $\{0.3,0.4\}$ | $\{0.4,0.5\}$ | $\{0.8\}$ |

To get the optimal supplier, the following steps are given:
Step 1. Calculate the weights $w_{i j}^{(k)}(k=1,2,3)$ associated with the evaluated values $h_{i j}^{(k)}(k=1,2,3)$ :

$$
V^{(1)}=\left(w_{i j}^{(1)}\right)_{3 \times 3}=\left(\begin{array}{lll}
0.3333 & 0.3160 & 0.3393 \\
0.3333 & 0.3273 & 0.3333 \\
0.3319 & 0.3304 & 0.3364
\end{array}\right)
$$

$$
\begin{aligned}
& V^{(2)}=\left(w_{i j}^{(2)}\right)_{3 \times 3}=\left(\begin{array}{lll}
0.3241 & 0.3443 & 0.3348 \\
0.3333 & 0.3364 & 0.3377 \\
0.3362 & 0.3348 & 0.3364
\end{array}\right) \\
& V^{(3)}=\left(w_{i j}^{(3)}\right)_{3 \times 3}=\left(\begin{array}{lll}
0.3426 & 0.3396 & 0.3259 \\
0.3333 & 0.3364 & 0.3289 \\
0.3319 & 0.3348 & 0.3273
\end{array}\right)
\end{aligned}
$$

Step 2. Assume that the weights of the attributes have correlations with each other and

$$
\dot{m}(\phi)=0, \dot{m}\left(\left\{x_{1}\right\}\right)=0.3, \dot{m}\left(\left\{x_{2}\right\}\right)=0.5, \dot{m}\left(\left\{x_{3}\right\}\right)=0.4
$$

then we have

$$
\begin{gathered}
\dot{m}\left(\left\{x_{1}, x_{2}\right\}\right)=0.7323, \dot{m}\left(\left\{x_{1}, x_{3}\right\}\right)=0.6458 \\
\dot{m}\left(\left\{x_{2}, x_{3}\right\}\right)=0.8097, \dot{m}\left(\left\{x_{1}, x_{2}, x_{3}\right\}\right)=1
\end{gathered}
$$

Step 3. Utilize Eq.(1.203) (let $\phi(\gamma)=\gamma)$ to aggregate all the individual hesitant fuzzy decision matrices $H_{k}=\left(h_{i j}^{(k)}\right)_{3 \times 3}(k=1,2,3)$ into the collective hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{3 \times 3}$ (see Table 1.11 (Xia et al. 2013a)).

Table 1.11. The collective hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.4361,0.5009\}$ | $\{0.4264,0.4604,0.4953$, | $\{0.4652,0.4991\}$ |
|  |  | $0.5292\}$ |  |
| $A_{2}$ | $\{0.7000,0.7667$, | $\{0.6682,0.7991\}$ | $\{0.6004,0.6333,0.6342$, |
|  | $0.8333\}$ |  | $0.6671\}$ |
| $A_{3}$ | $\{0.3668,0.4000$, | $\{0.3335,0.3670,0.4339$, | $\{0.6318,0.6991\}$ |
|  | $0.4332\}$ | $0.4674\}$ |  |

Step 4. Utilize Eq.(1.207) (let $g(\gamma)=\gamma$ ) to get the expected results $h_{i}(i=1,2,3)$ for the alternatives $A_{i}(i=1,2,3)$ :

$$
\begin{aligned}
h_{1}= & \{0.4397,0.4488,0.4544,0.4591,0.4635,0.4682,0.4695,0.4738, \\
& 0.4786,0.4829,0.4841,0.4889,0.4932,0.4980,0.5036,0.5127\} \\
h_{2}= & \{0.6596,0.6684,0.6686,0.6774,0.6796,0.6884,0.6886,0.6975,0.6996, \\
& 0.7084,0.7086,0.7162,0.7174,0.7250,0.7252,0.7340,0.7362,0.7450, \\
& 0.7452,0.7540,0.7562,0.7650,0.7652,0.7740\} \\
h_{3}= & \{0.4233,0.4333,0.4378,0.4414,0.4433,0.4478,0.4513,0.4558,0.4577, \\
& 0.4613,0.4658,0.4667,0.4758,0.4767,0.4812,0.4848,0.4867,0.4912 \\
& 0.4947,0.4992,0.5011,0.5047,0.5092,0.5192\}
\end{aligned}
$$

Step 5. Calculate the scores of $h_{i}(i=1,2,3)$, and rank the alternatives according to $s\left(h_{i}\right)(i=1,2,3)$ :

$$
h_{1}=0.4762, h_{2}=0.7168, h_{3}=0.4713
$$

and thus $A_{2} \succ A_{1} \succ A_{3}$.
In Step 3, if we utilize Eq.(1.204) (let $\left.\phi_{i}(\gamma)=\gamma, i=1,2,3\right)$ to aggregate all the individual hesitant fuzzy decision matrices $H_{k}=\left(h_{i j}^{(k)}\right)_{3 \times 3}(k=1,2,3)$ into the collective hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{3 \times 3}$ (see Table 1.12 (Xia et al. 2013a)).

Table 1.12. The collective hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.3948,0.4943\}$ | $\{0.3921,0.4324,0.4675$ | $\{0.4565,0.4924\}$ |
|  |  | $, 0.5155\}$ |  |
| $A_{2}$ | $\{0.6952,0.7560,0.7652$ | $\{0.6558,0.7949\}$ | $\{0.5949,0.6267$, |
|  | $, 0.8320\}$ |  | $0.6316,0.6654\}$ |
| $A_{3}$ | $\{0.3636,0.3915,0.4000$ | $\{0.3303,0.3560,0.4166$ | $\{0.6200,0.6943\}$ |
|  | $, 0.4307\}$ | $, 0.4489\}$ |  |
|  |  |  |  |

We first utilize Eq.(1.207) (let $\left.\phi_{i}(\gamma)=\gamma, i=1,2,3\right)$ to get the expected results $h_{i}(i=1,2,3)$ for the alternatives $A_{i}(i=1,2,3)$ :

$$
\begin{aligned}
& h_{1}=\{0.4092,0.4176,0.4269,0.4357,0.4378,0.4416,0.4467,0.4506, \\
&0.4567,0.4606,0.4660,0.4700,0.4724,0.4820,0.4927,0.5028\}
\end{aligned}
$$

$$
h_{2}=\{0.6502,0.6593,0.6607,0.6668,0.6692,0.6700,0.6761,0.6775,
$$

0.6786,0.6800,0.6862,0.6870,0.6895,0.6958,0.6973,0.7066,

$$
0.7071,0.7165,0.7180,0.7246,0.7272,0.7281,0.7347,0.7363
$$

$$
0.7374,0.7390,0.7457,0.7466,0.7493,0.7562,0.7577,0.7684\}
$$

$h_{3}=\{0.4024,0.4114,0.4141,0.4148,0.4156,0.4234,0.4241,0.4249$, $0.4268,0.4277,0.4284,0.4364,0.4373,0.4380,0.4409,0.4449$, $0.4507,0.4548,0.4578,0.4585,0.4594,0.4680,0.4688,0.4698$, $0.4719,0.4728,0.4736,0.4824,0.4834,0.4842,0.4873,0.4983\}$

Then we calculate the scores of $h_{i}(i=1,2,3)$, and rank the alternatives according to $s\left(h_{i}\right)(i=1,2,3)$ :

$$
h_{1}=0.4543, h_{2}=0.7076, h_{3}=0.4485
$$

and thus, $A_{2} \succ A_{1} \succ A_{3}$.
In the decision making process, we can choose different aggregation operators according to the practical problems. At the same time, different results may be produced, which reflects the preferences of the DMs.

Motivated by Definitions 1.35 and 1.36 , if we replace the arithmetical average and the arithmetical geometric average in Definitions 1.22 and 1.23 with the quasi arithmetical average, respectively, then the QHFHAA and QHFHAG operators will be obtained, which are in mathematical forms as below:

Definition 1.43 (Liao and Xu 2013c). For a collection of HFEs $h_{j}(j=1,2, \cdots, n), \quad w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of them with $w_{j} \in[0,1], \quad j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$, then we define the following
aggregation operators, which are all based on the mapping $\Theta^{n} \rightarrow \Theta$ with an aggregation-associated vector $\omega=\left(\omega_{1}, \omega_{2}, \cdots, \omega_{n}\right)^{\mathrm{T}}$ such that $\omega_{j} \in[0,1]$, $j=1,2, \ldots, n, \sum_{j=1}^{n} \omega_{j}=1$, and a continuous strictly monotonic function $g(\gamma)$ :
(1) The quasi hesitant fuzzy hybrid arithmetical averaging (QHFHAA) operator:
$\operatorname{QHFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=g^{-1}\left(\frac{\bigoplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} g\left(h_{j}\right)}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{g^{-1}\left(1-\prod_{j=1}^{n}\left(1-g\left(\gamma_{j}\right)\right)^{\frac{\lambda_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}}\right)\right\} \tag{1.211}
\end{equation*}
$$

(2) The quasi hesitant fuzzy hybrid arithmetical geometric (QHFHAG) operator:
$\operatorname{QHFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=g^{-1}\left(\underset{j=1}{n}\left(g\left(h_{j}\right)\right)^{\frac{w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{g^{-1}\left(\prod_{j=1}^{n}\left(g\left(\gamma_{j}\right)\right)^{\frac{\lambda_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} \lambda_{j} \omega_{\sigma(j)}}}\right)\right\} \tag{1.212}
\end{equation*}
$$

where $\sigma:\{1,2, \cdots, n\} \rightarrow\{1,2, \cdots, n\}$ is the permutation such that $h_{j}$ is the $\sigma(j)$ th largest element of the collection of HFEs $h_{j}(j=1,2, \cdots, n)$, and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weighting vector of the HFEs $h_{j}(j=1,2, \cdots, n)$, with $w_{j} \in[0,1], j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$.

Note that when assigning different weighting vector of $\omega$ or $w$ or choosing different types of function $g(\gamma)$, the QHFHAA and QHFHAG operators will reduce to many special cases, which can be set out as follows:
(1) If the associated weighting vector $\omega=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the QHFHAA operator reduces to the QHFWAA operator shown as:

$$
\begin{align*}
\operatorname{QHFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) & =g^{-1}\left(\bigoplus_{j=1}^{n} w_{j} g\left(h_{j}\right)\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{g^{-1}\left(1-\prod_{j=1}^{n}\left(1-g\left(\gamma_{j}\right)\right)^{w_{j}}\right)\right\} \tag{1.213}
\end{align*}
$$

while the QHFHAG operator reduces to the QHFWG operator shown as:
$\operatorname{QHFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=g^{-1}\left(\underset{\left.\left.\underset{j=1}{n}\left(g\left(h_{j}\right)\right)^{w_{j}}\right)\right) ~}{x}\right.$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{g^{-1}\left(\prod_{j=1}^{n}\left(g\left(\gamma_{j}\right)\right)^{w_{j}}\right)\right\} \tag{1.214}
\end{equation*}
$$

(2) If the arguments' weight vector $w=\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^{T}$, then the QHFHAA operator reduces to the QHFOWA operator given as Definition 1.35, while the QHFHAG operator reduces to the QHFOWG operator given as Definition 1.36 .
(3) If $g(\gamma)=\gamma$, then the QHFHAA operator reduces to the HFHAA operator given as Definition 1.22, while the QHFHAG operator reduces to the HFHAG operator given as Definition 1.23. It is obvious and herein we don't show some proofs.
(4) If $g(\gamma)=\ln \gamma$, then the QHFHAA operator reduces to the HFHAG operator given as Definition 1.23, while the QHFHAG operator reduces to the HFHAA operator given as Definition 1.22. The derivation can be shown as below:
$\operatorname{QHFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=e^{\frac{\substack{\oplus \\ j=1 \\ \sum_{j} \omega_{\sigma(j)} \ln \left(h_{j}\right)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}=\left(e^{\substack{\oplus=1 \\ j=1 \\ j}} \omega_{\sigma(j)} \ln \left(h_{j}\right)\right)^{\frac{1}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}}$

$$
\begin{equation*}
=\bigotimes_{j=1}^{n}\left(h_{j}\right)^{\frac{\lambda_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{\lambda_{j} \omega_{\sigma(j)}}}}=\operatorname{HFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.215}
\end{equation*}
$$

while

$$
\begin{align*}
& =\frac{\bigoplus_{j=1}^{n} w_{j} \omega_{\sigma(j)} h_{j}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}=\operatorname{HFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right) \tag{1.216}
\end{align*}
$$

Some other special cases can also be constructed by choosing different types of the function $g(\gamma)$ for the QHFHAA and QHFHAG operators, such as

$$
\begin{gather*}
g(\gamma)=\gamma^{\lambda}, g(\gamma)=1-(1-\gamma)^{\lambda}, g(\gamma)=\sin \left(\frac{\pi}{2} \gamma\right), g(\gamma)=1-\sin \left(\frac{\pi}{2}(1-\gamma)\right)  \tag{1.217}\\
g(\gamma)=\cos \left(\frac{\pi}{2} \gamma\right), g(\gamma)=1-\cos \left(\frac{\pi}{2}(1-\gamma)\right), g(\gamma)=\tan \left(\frac{\pi}{2} \gamma\right)  \tag{1.218}\\
g(\gamma)=1-\tan \left(\frac{\pi}{2}(1-\gamma)\right), g(\gamma)=\lambda^{\gamma}, g(\gamma)=1-b^{1-\gamma} \tag{1.219}
\end{gather*}
$$

where $\lambda, b>0$ and $\lambda, b \neq 1$.
In the following, we try to investigate the properties of the QHFHAA and QHFHAG operators:

Theorem 1.43 (Idempotency) (Liao and Xu 2013c). If $h_{j}=h(j=1,2, \cdots, n)$, then

$$
\begin{equation*}
\operatorname{QHFHAA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h, \operatorname{QHFHAG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=h \tag{1.220}
\end{equation*}
$$

Proof. According to Definition 1.43, we can obtain

$$
\begin{align*}
& =g^{-1}\left(\frac{g(h) \sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}{\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}}\right)=g^{-1}(g(h))=h \tag{1.221}
\end{align*}
$$

$$
\begin{align*}
& =g^{-1}\left((g(h))^{\substack{\begin{subarray}{c}{\sum_{i=1}^{n} w_{j} \omega_{\sigma(j)} \\
\sum_{j=1}^{n} w_{j} \omega_{\sigma(j)}} }}\end{subarray}}\right)=g^{-1}(g(h))=h \tag{1.222}
\end{align*}
$$

which completes the proof of the theorem.
Consider a group decision making problem under uncertainty. Suppose that the DM $D_{k} \in D$ provides all the possible evaluated values under the attribute $x_{j} \in X$ for the alternative $A_{i}$ denoted by a HFE $h_{i j}^{(k)}$ and constructs the decision matrix $H_{k}=\left(h_{i j}^{(k)}\right)_{n \times m}$. He/She also determines the importance degrees $w_{j}^{(k)}(j=1,2, \ldots, m)$ for the relevant attributes according to his/her preferences. Meanwhile, since different alternatives may have different focuses and advantages, to reflect this issue, the DM also gives the ordering weights $\omega_{j}^{(k)}(j=1,2, \ldots, n)$ for different attributes. Suppose that the weight vector of the DMs is $v=\left(v_{1}, v_{2}, \ldots, v_{p_{0}}\right)^{\mathrm{T}}, v_{k} \in[0,1], k=1,2, \ldots, p_{0}$, and $\sum_{k=1}^{p_{0}} v_{k}=1$. Then, based on the given aggregation operators, we introduce a method for group
decision making with hesitant fuzzy information, which involves the following steps (Liao and Xu 2013c):

Step 1. Utilize the HFWA operator (1.32) (or the HFWG operator (1.34)) to aggregate all the individual hesitant fuzzy decision matrix $H_{k}=\left(h_{i j}^{(k)}\right)_{n \times m}$ $\left(k=1,2, \ldots, p_{0}\right)$ into the collective hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{n \times m}$, where

$$
\begin{equation*}
h_{i j}=\bigcup_{\substack{\gamma_{i j}^{(k)} \in h_{i j}^{(k)}, k=1,2, \cdots, p_{0}}}\left\{1-\prod_{k=1}^{p_{0}}\left(1-\gamma_{i j}^{(k)}\right)^{\lambda_{k}}\right\}, i=1,2, \ldots, n ; j=1,2, \ldots, m \tag{1.223}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i j}=\bigcup_{\gamma_{i j}^{(k)} \in h_{i j}^{(k)}, k=1,2, \cdots, p_{0}}\left\{\prod_{k=1}^{p_{0}}\left(\gamma_{i j}^{(k)}\right)^{\lambda_{k}}\right\}, i=1,2, \ldots, n ; j=1,2, \ldots, m \tag{1.224}
\end{equation*}
$$

Step 2. Utilize the QHFHAA (or QHFHAG) operator to obtain the HFEs $h_{i}$ $(i=1,2, \cdots, n)$ for the alternatives $A_{i}(i=1,2, \cdots, n)$, where

$$
\begin{align*}
h_{i} & =\operatorname{QHFHAA}\left(h_{i 1}, h_{i 2}, \cdots, h_{i m}\right) \\
& =\bigcup_{\gamma_{i j} \in h_{i j}, j=1,2, \cdots, m}\left\{g^{-1}\left(1-\prod_{j=1}^{m}\left(1-g\left(\gamma_{j}\right)\right)^{\frac{w_{j} \omega_{e(j)}}{\sum_{j=1}^{m} w_{j} \omega_{e(j)}}}\right)\right\}, i=1,2, \cdots, n \tag{1.225}
\end{align*}
$$

or

$$
\begin{align*}
h_{i}= & \operatorname{QHFHAG}\left(h_{i 1}, h_{i 2}, \cdots, h_{i m}\right) \\
& =\bigcup_{\gamma_{i j} \in h_{i j}, j=1,2, \cdots, m}\left\{g^{-1}\left(\prod_{j=1}^{m}\left(g\left(\gamma_{j}\right)\right)^{\substack{\sum_{j=1}^{m} \lambda_{j(j)} \omega_{\sigma(j)}}}\right)\right\}, i=1,2, \cdots, n \tag{1.226}
\end{align*}
$$

Step 3. Compute the score values $s\left(h_{i}\right)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ by Definition 1.2 and the deviation degrees $\bar{\sigma}^{\prime}(h)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ by Definition 1.5.

Step 4. Get the priority of the alternatives $A_{i}(i=1,2, \cdots, n)$ by ranking $s\left(h_{i}\right)$ and $\bar{\sigma}^{\prime}(h)(i=1,2, \cdots, n)$.

We now consider a multi-attribute group decision making problem that concerns evaluating and ranking work systems safety (adapted from Dağdeviren and Yüksel (2008)) to illustrate our method:

Example 1.23 (Liao and Xu 2013c). Maintaining the safety of work systems in workplace is one of the most important components of safety management within an effective manufacturing organization. There are many factors which affect the safety system simultaneously. According to the statistical analysis of the past work accidents in a manufacturing company in Ankara, Turkey, Dağdeviren \& Yüksel (2008) found there are four sorts of factors which affect the safety system:
(1) $x_{1}$ : Organizational factors, which involve job rotation, working time, job completion pressure, and insufficient control.
(2) $x_{2}$ : Personal factors, which consist of insufficient preparation, insufficient responsibility, tendency of risky behavior, and lack of adaptation.
(3) $x_{3}$ : Job related factors, which can be divided into job related fatigue, reduced operation times due to dangerous behaviors, and variety and dimension of job related information.

In addition, it is not possible to assume that the effects of all factors of work safety are the same in all cases. Hence, by using the fuzzy analytic hierarchy process (FAHP) method, Dağdeviren and Yüksel (2008) constructed a hierarchical structure to depict the factors and sub-factors, and then determined the weight vector of these three factors, which is $w=(0.388,0.3,0.312)^{\mathrm{T}}$. Three DMs $D_{k}(k=1,2,3)$ from different departments, whose weight vector is $v=(0.4,0.3,0.3)^{\mathrm{T}}$, are gathered together to evaluate three candidate work systems $A_{i}(i=1,2,3)$ according to the above predetermined factors $x_{j}(j=1,2,3,4)$. However, since these factors effecting work system safety have non-physical structures, it is hard for the DMs to represent their preference by using crisp numbers. HFEs are appropriate for them to use in expressing these preferences realistically since they may have a set of possible values when evaluating these behavioral and qualitative factors. Thus, the hesitant fuzzy judgment matrices $H_{k}=\left(h_{i j}^{(k)}\right)_{3 \times 4}(k=1,2,3)$ are constructed by the DMs , shown as Tables 1.13-1.15 (Liao and Xu 2013c). Furthermore, considering the fact that different DMs are familiar with different research fields, and meanwhile, different work systems may focus on different partitions,
the DMs may want to give more weights to the attribute which is more prominent. Hence, another weight vectors are determined by the DMs according to their preferences, which are $\omega^{(1)}=(0.4,0.3,0.3)^{\mathrm{T}}, \omega^{(2)}=(0.5,0.3,0.2)^{\mathrm{T}}$ and $\omega^{(3)}=(0.4,0.4,0.2)^{\mathrm{T}}$.

Table 1.13. The hesitant fuzzy decision matrix $H_{1}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.6\}$ | $\{0.7\}$ | $\{0.4,0.5\}$ |
| $A_{2}$ | $\{0.6,0.8\}$ | $\{0.5,0.9\}$ | $\{0.7\}$ |
| $A_{3}$ | $\{0.4,0.5\}$ | $\{0.3\}$ | $\{0.6\}$ |

Table 1.14. The hesitant fuzzy decision matrix $\mathrm{H}_{2}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4\}$ | $\{0.3,0.5\}$ | $\{0.4\}$ |
| $A_{2}$ | $\{0.8\}$ | $\{0.7\}$ | $\{0.6,0.7\}$ |
| $A_{3}$ | $\{0.4\}$ | $\{0.3,0.6\}$ | $\{0.5,0.7\}$ |

Table 1.15. The hesitant fuzzy decision matrix $\mathrm{H}_{3}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5\}$ | $\{0.3,0.4\}$ | $\{0.6\}$ |
| $A_{2}$ | $\{0.7,0.9\}$ | $\{0.8\}$ | $\{0.5,0.6\}$ |
| $A_{3}$ | $\{0.3,0.4\}$ | $\{0.4,0.5\}$ | $\{0.8\}$ |

To get the optimal work system, the following steps are given:
Step 1. Utilize the aggregation operator to fuse all the individual hesitant fuzzy decision matrices $H_{k}=\left(h_{i j}^{(k)}\right)_{3 \times 4}(k=1,2,3)$ into the collective hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{3 \times 4}$. Here we use the HFWA operator (1.223) to fuse the individual hesitant fuzzy decision matrix. Here, we take $h_{23}$ as an example:

$$
\begin{aligned}
h_{23} & =\bigcup_{\gamma_{23}^{(k)} \in h_{23}^{k(k)}, k=1,2,3}\left\{1-\prod_{k=1}^{3}\left(1-\gamma_{23}^{(k)}\right)^{\lambda_{k}}\right\} \\
= & \left\{1-(1-0.7)^{0.4}(1-0.6)^{0.3}(1-0.5)^{0.3}, 1-(1-0.7)^{0.4}(1-0.6)^{0.3}(1-0.6)^{0.3},\right. \\
& \left.1-(1-0.7)^{0.4}(1-0.7)^{0.3}(1-0.5)^{0.3}, 1-(1-0.7)^{0.4}(1-0.7)^{0.3}(1-0.6)^{0.3}\right\} \\
& =\{0.619,0.644,0.65,0.673\}
\end{aligned}
$$

Similarly, other fused values can be obtained, and then the collective hesitant fuzzy matrix can be derived as below:
$H=\left(\begin{array}{ccc}\{0.473,0.517\} & \{0.501,0.524,0.549,0.57\} & \{0.469,0.506\} \\ \{0.702,0.774,0.786,0.838\} & \{0.674,0.829\} & \{0.619,0.644,0.65,0.673\} \\ \{0.372,0.4,0.442,0.416\} & \{0.332,0.367,0.435,0.465\} & \{0.653,0.702\}\end{array}\right)$
Step 2. Utilize the aggregation operator (such as the QHFHAA operator (1.225) or the QHFHAG operator (1.226)) to obtain the HFEs $h_{i}(i=1,2,3)$ for the alternatives $A_{i}(i=1,2,3)$. Here we use the QHFHAA operator to fuse the collective HFEs and let $g(\gamma)=\gamma$, then we can get

$$
\begin{aligned}
h_{1}= & \{0.4825,0.4913,0.493,0.4983,0.5012,0.5017,0.5068,0.5084,0.5098, \\
& 0.5113,0.5164,0.5168,0.5197,0.5247,0.5262,0.5343\} \\
h_{2}= & \{0.6811,0.685,0.686,0.6898,0.7269,0.7302,0.7303,0.731,0.7336,0.7343, \\
& 0.7344,0.7351,0.7383,0.7391,0.7377,0.7423,0.769,0.7718,0.7725, \\
& 0.7733,0.7753,0.776,0.7761,0.7768,0.7787,0.7794,0.7795,0.7821, \\
& 0.8083,0.8106,0.8112,0.8135\}
\end{aligned}
$$

$$
\begin{aligned}
h_{3}= & \{0.4894,0.4942,0.4999,0.5043,0.5046,0.506,0.509,0.5107,0.5145, \\
& 0.5162,0.5171,0.5191,0.5204,0.5208,0.5217,0.525,0.5271,0.5303, \\
& 0.5312,0.5316,0.5329,0.5348,0.5357,0.5373,0.5409,0.5425,0.5453, \\
& 0.5465,0.5468,0.5508,0.5558,0.5601\}
\end{aligned}
$$

The computational process of $h_{2}$ can be illustrated as an example:
Since

$$
\begin{gathered}
s\left(h_{21}\right)=\frac{(0.702+0.774+0.786+0.838)}{4}=0.775 \\
s\left(h_{22}\right)=\frac{(0.674+0.829)}{2}=0.7515 \\
s\left(h_{23}\right)=\frac{(0.619+0.644+0.65+0.673)}{4}=0.6465
\end{gathered}
$$

then $h_{21}>h_{22}>h_{23}$. Thus, $\sigma(21)=1, \sigma(22)=2, \sigma(23)=3$, and

$$
\begin{gathered}
\frac{w_{1} \omega_{\sigma(21)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=\frac{0.388 \times 0.5}{0.388 \times 0.5+0.3 \times 0.3+0.312 \times 0.2}=0.56 \\
\frac{w_{2} \omega_{\sigma(22)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=0.2598, \frac{w_{3} \omega_{\sigma(23)}}{\sum_{j=1}^{3} w_{j} \omega_{\sigma(2 j)}}=0.1801
\end{gathered}
$$

Therefore, by using Eq.(1.225), we can calculate that

$$
\begin{aligned}
& h_{2}=\operatorname{QHFHAA}\left(h_{21}, h_{22}, h_{23}\right) \\
&= \bigcup_{\gamma_{21} \in h_{21}, \gamma_{22} \in h_{22}, \gamma_{23} \in h_{23}}\left\{1-\left(1-\gamma_{21}\right)^{0.56}\left(1-\gamma_{22}\right)^{0.2598}\left(1-\gamma_{23}\right)^{0.1801}\right\} \\
&=\{0.6811,0.685,0.686,0.6898,0.7269,0.7302,0.7303,0.731,0.7336,0.7343, \\
& 0.7344,0.7351,0.7383,0.7391,0.7377,0.7423,0.769,0.7718,0.7725,
\end{aligned}
$$

$0.7733,0.7753,0.776,0.7761,0.7768,0.7787,0.7794,0.7795,0.7821$, $0.8083,0.8106,0.8112,0.8135\}$

Step 3. Compute the score values $s\left(h_{i}\right)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ by Definition 1.2, and then we have $s\left(h_{1}\right)=0.5089, s\left(h_{2}\right)=0.7534$, and $s\left(h_{3}\right)=0.5257$.
Step 4. Since $s\left(h_{2}\right)>s\left(h_{3}\right)>s\left(h_{1}\right)$, then we get $h_{2} \succ h_{3} \succ h_{1}$, which means $A_{2}$ is the most desirable work system.

### 1.6 Generalized Hesitant Fuzzy Aggregation

It is noted that the known hesitant fuzzy aggregation operators are all developed based on the Algebraic $t$-norm and $t$-conorm which is a special case of $t$-norms and $t$-conorms. In this section, we mainly introduce some aggregation operators for hesitant fuzzy information, discuss their properties and special cases, and give their applications to MADM.

Definition 1.44 (Xia and Xu 2011b). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ their weight vector $w_{i} \in[0,1], i=1,2, \cdots, n$, and $\sum_{i=1}^{n} w_{i}=1$. If

$$
\begin{equation*}
\text { ATS-HFWA }\left(h_{1}, h_{2}, \cdots, h_{n}\right)={ }_{i=1}^{n} w_{i} h_{i}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right\} \tag{1.227}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { ATS-HFWG }\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\stackrel{n}{\otimes} h_{i=1}^{w_{i}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{\tau}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{\tau}\left(\gamma_{i}\right)\right)\right\} \tag{1.228}
\end{equation*}
$$

then ATS-HFWA and ATS-HFWG are called the Archimedean t-norm and tconorm based hesitant fuzzy weighted averaging (ATS-HFWA) operator and the Archimedean $t$-norm and $t$-conorm based hesitant fuzzy weighted geometric (ATS-HFWG) operator, respectively.

Now we show that Eq.(1.227) holds, by using mathematical induction on $n$ : For $n=2$, we have
$\operatorname{ATS}-\operatorname{HFWA}\left(h_{1}, h_{2}\right)=\stackrel{2}{\oplus}{ }_{i=1} w_{i} h_{i}=w_{1} h_{1} \oplus w_{2} h_{2}$

$$
\begin{align*}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(w_{1} \dot{s}\left(\gamma_{1}\right)\right)\right)+\dot{s}\left(\dot{s}^{-1}\left(w_{2} \dot{s}\left(\gamma_{2}\right)\right)\right)\right)\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\dot{s}^{-1}\left(w_{1} \dot{s}\left(\gamma_{1}\right)+w_{2} \dot{s}\left(\gamma_{2}\right)\right)\right\} \tag{1.229}
\end{align*}
$$

Suppose it holds for $n=k$, that is,

$$
\begin{array}{r}
\operatorname{ATS-HFWA}\left(h_{1}, h_{2}, \cdots, h_{k}\right)=\stackrel{k}{i=1} w_{i} h_{i}=w_{1} h_{1} \oplus w_{2} h_{2} \oplus \cdots \oplus w_{k} h_{k} \\
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{k} \in h_{k}}\left\{\dot{s}^{-1}\left(\sum_{i=1}^{k} w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right\} \tag{1.230}
\end{array}
$$

then

$$
\begin{align*}
& \text { ATS-HFWA }\left(h_{1}, h_{2}, \cdots, h_{k}, h_{k+1}\right)=\left(\oplus_{i=1}^{k} w_{i} h_{i}\right) \oplus w_{k+1} h_{k+1} \\
& =\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{k} \in h_{k}}\left\{\dot{s}^{-1}\left(\sum_{i=1}^{k} w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right\}\right) \oplus\left(\bigcup_{\gamma_{k+1} \in h_{k+1}}\left\{\dot{s}^{-1}\left(w_{k+1} \dot{s}\left(\gamma_{k+1}\right)\right)\right\}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{k} \in h_{k}, \gamma_{k+1} \in h_{k+1}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(\sum_{i=1}^{k} w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right)+\dot{s}\left(\dot{s}^{-1}\left(w_{k+1} \dot{s}\left(\gamma_{k+1}\right)\right)\right)\right)\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{k} \in h_{k}, \gamma_{k+1} \in h_{k+1}}\left\{\dot{s}^{-1}\left(\sum_{i=1}^{k} w_{i} \dot{s}\left(\gamma_{i}\right)+w_{k+1} \dot{s}\left(\gamma_{k+1}\right)\right)\right\} \\
& =  \tag{1.231}\\
& \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{k} \in h_{k}, \gamma_{k+1} \in h_{k+1}}\left\{\dot{s}^{-1}\left(\sum_{i=1}^{k+1} w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right\}
\end{align*}
$$

Similarly, we can prove that Eq.(1.227) holds, which completes the proof.
Definition 1.45 (Xia and Xu 2011b). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ the weight vector of them with $w_{i} \in[0,1]$, $i=1,2, \cdots, n, \sum_{i=1}^{n} w_{i}=1$ and $\lambda>0$. If
$\operatorname{ATS}-\operatorname{GHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\left(\bigoplus_{i=1}^{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}}$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\tau}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right)\right)\right)\right\}\right. \tag{1.232}
\end{equation*}
$$

and
$\operatorname{ATS}-\operatorname{GHFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\frac{1}{\lambda}\left(\underset{i=1}{n} \lambda h_{i}^{w_{i}}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{s}^{-1}\left(\frac{1}{\lambda} \dot{s}\left(\dot{\tau}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{\tau}\left(s^{-1}\left(\lambda \dot{s}\left(\gamma_{i}\right)\right)\right)\right)\right)\right\}\right. \tag{1.233}
\end{equation*}
$$

then ATS-GHFWA and ATS-GHFWG are called the Archimedean t-norm and tconorm based generalized hesitant fuzzy weighted averaging (ATS-GHFWG) operator and the Archimedean t-norm and t-conorm based generalized hesitant fuzzy weighted geometric (ATS-GHFWG) operator, respectively. Especially, if $\lambda=1$, then the ATS-GHFWA and ATS-GHFWG operators reduce to the ATSHFWA and ATS-HFWG operator, respectively.

We can show Eq.(1.232) holds, in fact, based on the operations for HFEs defined in Definition 1.13, we have $h_{i}^{\lambda}=\bigcup_{\gamma_{i} \in h_{i}}\left\{\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right\}$, then

$$
\begin{equation*}
\oplus_{i=1}^{n} w_{i} h_{i}^{\lambda}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{s}^{-1}\left(\oplus_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right)\right)\right\} \tag{1.234}
\end{equation*}
$$

and

ATS-GHFWA $\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\underset{\gamma_{\in} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}{\bigcup}\left\{\dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\tau}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right)\right)\right)\right)\right\}$

Similarly, we can prove that Eq.(1.232) also holds, which completes the proof.
Some properties of the ATS-GIFWA and ATS-GIFWG operators can be discussed as follows:

Theorem 1.44 (Xia and Xu 2011b). If all $h_{i}(i=1,2, \ldots, n)$ are equal, i.e., $h_{i}=h$, for all $i$, then

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHFWA}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=h \tag{1.236}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=h \tag{1.237}
\end{equation*}
$$

Proof. Let $h_{i}=h$, we have

ATS-GHFWA $\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\operatorname{ATS-GHFWA}(h, h, \ldots, h)$

$$
\begin{equation*}
=\bigoplus_{i=1}^{n} w_{i} h=\bigcup_{\gamma \in h}\left\{\dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\tau}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}(\lambda \dot{\tau}(\gamma))\right)\right)\right)\right\}=h\right. \tag{1.238}
\end{equation*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.

Theorem 1.45 (Xia and Xu 2011b). Let $h_{i}$ and $h_{i}^{*}(i=1,2, \ldots, n)$ be two collections of HFEs, if $\gamma_{i} \leq \gamma_{i}^{*}$ for all $\gamma_{i} \in h_{i}$ and $\gamma_{i}^{*} \in h_{i}^{*}$, then

$$
\begin{equation*}
\text { ATS-GHFWA }\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq \operatorname{ATS}-\operatorname{GHFWA}\left(h_{1}^{*}, h_{2}^{*}, \ldots, h_{n}^{*}\right) \tag{1.239}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { ATS-GHFWA }\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq \operatorname{ATS}-\operatorname{GHFWA}\left(h_{1}^{*}, h_{2}^{*}, \ldots, h_{n}^{*}\right) \tag{1.240}
\end{equation*}
$$

Proof. We have known that $\dot{s}(t)=\dot{\tau}(1-t)$, and $\dot{\tau}:[0,1] \rightarrow[0,+\infty]$ is a strictly decreasing function, then $\dot{s}(t)$ is a strictly increasing function. Since $\gamma_{i} \leq \gamma_{i}^{*}$ for all $\gamma_{i} \in h_{i}$ and $\gamma_{i}^{*} \in h_{i}^{*}$, then we have

$$
\begin{equation*}
\dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\lambda}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right)\right)\right)\right) \leq \dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\lambda}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}^{*}\right)\right)\right)\right)\right)\right) \tag{1.241}
\end{equation*}
$$

then

$$
\begin{equation*}
s\left(\text { ATS-GHFWA }\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq \text { ATS-GHFWA }\left(h_{1}^{*}, h_{2}^{*}, \ldots, h_{n}^{*}\right)\right. \tag{1.242}
\end{equation*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.

Based on Theorem 1.41, the following property can be obtained:
Theorem 1.46 (Xia and Xu 2011b). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs, and $h^{+}=\bigcup_{\gamma_{i} \in h_{i}} \max _{i}\left\{\gamma_{i}\right\}, h^{-}=\bigcup_{\gamma_{i} \in h_{i}} \min _{i}\left\{\gamma_{i}\right\}$, then

$$
\begin{equation*}
h^{-} \leq \text {ATS-GHFWA }\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h^{+} \tag{1.243}
\end{equation*}
$$

and

$$
\begin{equation*}
h^{-} \leq \operatorname{ATS}-\operatorname{GHFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h^{+} \tag{1.244}
\end{equation*}
$$

If the additive generator $\dot{\tau}$ is assigned different forms, then some specific intuitionistic fuzzy aggregation operators can be obtained as follows:

Case 1. If $\dot{\tau}(t)=-\log (t)$, then the ATS-GHFWA and ATS-GIFWG operators reduce to the following:

$$
\begin{equation*}
\operatorname{GHFWA}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\left(1-\prod_{i=1}^{n}\left(\left(1-\gamma_{i}^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right. \tag{1.245}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{GHFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\left(1-\prod_{i=1}^{n}\left(\left(1-\left(1-\gamma_{i}\right)^{\lambda}\right)^{w_{i}}\right)^{\frac{1}{\lambda}}\right\}\right. \tag{1.246}
\end{equation*}
$$

which are the generalized hesitant fuzzy weighted averaging (GHFWA) operator and the generalized hesitant fuzzy weighted geometric (GHFWG) operator proposed by Xia and Xu (2011a).

Case 2. If $\dot{\tau}(t)=\log \left(\frac{2-t}{t}\right)$, then the ATS-GHFWA and ATS-GIFWG operators reduce to the following:

$$
\begin{equation*}
\operatorname{GEHFWA}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{2\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}{\left(\gamma^{+}+3 \gamma^{-}\right)^{\frac{1}{\lambda}}+\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.247}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\operatorname{GEHFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\left(\gamma^{+}+3 \gamma^{-}\right)^{\frac{1}{\lambda}}-\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}{\left(\gamma^{+}+3 \gamma^{-}\right)^{\frac{1}{\lambda}}+\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}\right)\right\} \tag{1.248}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma^{+}=\prod_{i=1}^{n}\left(\left(1+\left(1-\gamma_{i}\right)\right)^{\lambda}+3 \gamma_{i}^{\lambda}\right)^{w_{i}}, \gamma^{-}=\prod_{i=1}^{n}\left(\left(1+\left(1-\gamma_{i}\right)\right)^{\lambda}-\gamma_{i}^{\lambda}\right)^{w_{i}} \tag{1.249}
\end{equation*}
$$

which are the generalized Einstein hesitant fuzzy weighted averaging (GEHFWA) operator and the generalized Einstein hesitant fuzzy weighted geometric (GEHFWG) operator, respectively.

Case 3. If $\dot{\tau}(t)=\log \left(\frac{\zeta+(1-\zeta) t}{t}\right), \zeta>0$, then the ATS-GHFWA and ATS-GIFWG operators reduce to the following:

HHFWA $\left(h_{1}, h_{2}, \ldots, h_{n}\right)$
and
$\operatorname{HHFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\frac{\left(\gamma^{+}+\left(\zeta^{2}-1\right) \gamma^{-}\right)^{\frac{1}{\lambda}}-\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}{\left(\gamma^{+}+\left(\zeta^{2}-1\right) \gamma^{-}\right)^{\frac{1}{\lambda}}+(\gamma-1)\left(\gamma^{+}-\gamma^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.251}
\end{equation*}
$$

where

$$
\begin{gather*}
\gamma^{+}=\prod_{i=1}^{n}\left(\left(1+(\zeta-1)\left(1-\gamma_{i}\right)\right)^{\lambda}+\left(\zeta^{2}-1\right) \gamma_{i}^{\lambda}\right)^{w_{i}}  \tag{1.252}\\
\gamma^{-}=\prod_{i=1}^{n}\left(\left(1+(\zeta-1)\left(1-\gamma_{i}\right)\right)^{\lambda}-\gamma_{i}^{\lambda}\right)^{w_{i}} \tag{1.253}
\end{gather*}
$$

which are the generalized Hammer hesitant fuzzy weighted averaging (GHHFWA) operator and the generalized Hammer hesitant fuzzy weighted geometric (GHHFWG) operator, respectively. Especially, if $\zeta=1$, then the GHHFWA operator reduces to the GHFWA operator and the GHHFWG operator reduces to the GHFWG operator; If $\gamma=2$, then the GHHFWA operator reduces to the GEHFWA operator and the GHHFWG operator reduces to the GEHFWG operator.

Example 1.24 (Xia and Xu 2011 b ). Let $h_{1}=\{0.2,0.3\}, h_{2}=\{0.4\}$ and $h_{3}=$ $\{0.1,0.3,0.4\}$ be three HFEs, whose weight vector is $w=(0.2,0.1,0.7)^{\mathrm{T}}$, then we can use the ATS-GHIFWA or ATS-GHIFWG operator to aggregate them, without loss of generality, let $\dot{\tau}(t)=\log \left(\frac{\zeta+(1-\zeta) t}{t}\right)$, then we have

$$
\begin{gathered}
h_{A}=\operatorname{ATS}-\operatorname{GHFWA}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2583,0.2751,0.3098,0.3183,0.3844,0.3877\} \\
s\left(h_{A}\right)=0.3223
\end{gathered}
$$

$$
\begin{aligned}
h_{G}=\text { ATS-GHFWG }\left(h_{1}, h_{2}, h_{3}\right)= & \{0.1288,0.1375,0.2781,0.3068,0.3247,0.3713\} \\
& s\left(h_{G}\right)=0.2579
\end{aligned}
$$

As the values of the parameters $\zeta$ and $\lambda$ between 0 and 1 , the scores obtained by using the ATS-GHFWA and ATS-GHFWG operators are given in Figs. 1.8-1.9 (Xia and Xu 2011b), respectively.


Fig. 1.8. Scores obtained by the ATS-GHFWA operator


Fig. 1.9. Scores obtained by the ATS-GHFWG operator

It is noted that the scores obtained by the ATS-GHFWA operator are increasing as the value of the parameter $\lambda$ increases, while the scores obtained by the ATSGHFWG operator are quite the opposite. However, the scores obtained by the ATS-GHFWA operator are always bigger than the ones obtained by the ATSGHFWG operator.

Furthermore, we can discuss the relationships of the developed aggregation operators:

Theorem 1.47 (Xia and Xu 2011b). Let $h_{i}(i=1,2, \cdots, n)$ be a collection of HFEs with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1]$, $i=1,2, \ldots, n, \sum_{i=1}^{n} w_{i}=1$ and $\lambda>0$, then

(2) $\stackrel{n}{\otimes}_{i=1}^{\otimes}\left(h_{i}^{c}\right)^{w_{i}}=\left(\bigoplus_{i=1}^{n} w_{i} h_{i}\right)^{c}$.
(3) $\left(\bigoplus_{i=1}^{n} w_{i}\left(h_{i}^{c}\right)^{\lambda}\right)^{\frac{1}{\lambda}}=\left(\frac{1}{\lambda}\left(\stackrel{n}{\otimes}_{i=1}^{\otimes}\left(\lambda h_{i}\right)^{w_{i}}\right)\right)^{c}$.
(4) $\frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}\left(\lambda h_{i}^{c}\right)^{w_{i}}\right)=\left(\left({\left.\left.\underset{i=1}{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{c} \text {. } . ~ . ~ . ~}_{\text {in }}\right.\right.$

Proof. (1) $\oplus_{i=1}^{n} w_{i} h_{i}^{c}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{s}\left(w_{i} \dot{s}^{-1}\left(1-\gamma_{i}\right)\right)\right\}$

$$
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\dot{\tau}\left(w_{i} \dot{\tau}^{-1}\left(\gamma_{i}\right)\right)\right\}=\left(\bigotimes_{i=1}^{n} h_{i}^{w_{i}}\right)^{c} .
$$

(2) $\stackrel{n}{\otimes}_{i=1}^{\otimes}\left(h_{i}^{c}\right)^{w_{i}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{\tau}^{-1}\left(w_{i} \dot{\tau}\left(1-\gamma_{i}\right)\right)\right\}$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\dot{s}^{-1}\left(w_{i} \dot{s}\left(\gamma_{i}\right)\right)\right\}=\left(\bigoplus_{i=1}^{n} w_{i} h_{i}\right)^{c} . \\
& \text { (3) }\left(\bigoplus_{i=1}^{n} w_{i}\left(h_{i}^{c}\right)^{\lambda}\right)^{\frac{1}{\lambda}}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{\dot{\tau}^{-1}\left(\frac{1}{\lambda} \dot{\tau}\left(\dot{s}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(1-\gamma_{i}\right)\right)\right)\right)\right)\right\}\right. \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\dot{s}^{-1}\left(\frac{1}{\lambda} \dot{s}\left(\dot{\tau}^{-1}\left(\sum_{i=1}^{n} w_{i} \dot{\tau}\left(\dot{s}^{-1}\left(\lambda \dot{s}\left(\gamma_{i}\right)\right)\right)\right)\right)\right\}\right. \\
& =\left(\frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}\left(\lambda h_{i}\right)^{w_{i}}\right)\right)^{c} .
\end{aligned}
$$

$$
\begin{aligned}
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \cdots, \gamma_{n} \in h_{n}}\left\{1-\dot{\tau}^{-1}\left(\frac { 1 } { \lambda } \dot { \tau } \left(\dot { s } ^ { - 1 } \left({\left.\left.\left.\underset{i=1}{n} w_{i} \dot{s}\left(\dot{\tau}^{-1}\left(\lambda \dot{\tau}\left(\gamma_{i}\right)\right)\right)\right)\right)\right\}=\left(\left(\bigoplus_{i=1}^{n} w_{i} h_{i}^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{c} . . . ~ . ~ . ~ . ~}_{c}\right.\right.\right.\right.
\end{aligned}
$$

Suppose that there are $n$ alternatives $A_{i}(i=1,2, \cdots, n)$ and $m$ attributes $x_{j}(j=1,2, \cdots, m)$ with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}}$ such that $w_{j} \in[0,1], j=1,2, \cdots, n$, and $\sum_{j=1}^{n} w_{j}=1$. To get more reasonable results, a decision organization, which is constructed by a lot of DMs, is authorized to evaluate the alternatives under each attribute. For the alternative $A_{i}$ under the attribute $x_{j}$, some DMs in the decision organization may provide one value, others may provide another value, and both of these two parts can not persuade each other. To deal with such situation, HFEs are very useful tool, in which we consider all the possible values of the alternative $A_{i}$ under the attribute $x_{j}$ provided by the decision organization as a HFE $h_{i j}$. All the HFEs $h_{i j}(i=1,2, \cdots, n ; j=1,2, \cdots, m)$ are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{n \times m}$.

Based on the above analysis, we give the following decision making method (Xia and Xu 2011b):

Step 1. Utilize the developed hesitant fuzzy aggregation operators to obtain the HFEs $h_{i}(i=1,2, \cdots, n)$ for the alternatives $A_{i}(i=1,2, \cdots, n)$, i.e.,

$$
\begin{equation*}
h_{i}=\text { ATS-GHFWA }_{\lambda}\left(h_{i 1}, h_{i 2}, \cdots, h_{i n}\right)=\left(\stackrel{̣}{i=1}_{n}^{i_{j}} h_{i j}^{\lambda}\right)^{\frac{1}{\lambda}}, i=1,2, \cdots, n \tag{1.254}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i}=\operatorname{ATS}^{-G H F W G}{ }_{\lambda}\left(h_{i 1}, h_{i 2}, \cdots, h_{i n}\right)=\frac{1}{\lambda} \bigotimes_{i=1}^{n}\left(\lambda h_{i j}\right)^{w_{j}}, i=1,2, \cdots, m \tag{1.255}
\end{equation*}
$$

Step 2. Compute the score values $s\left(h_{i}\right)(i=1,2, \cdots, n)$ of $h_{i}(i=1,2, \cdots, n)$ and get the priority of the alternatives $A_{i}(i=1,2, \cdots, n)$ by ranking $s\left(h_{i}\right)$ $(i=1,2, \cdots, n)$.

Example 1.25 (Parreiras et al. 2010; Xia and Xu 2011b). An enterprise is to plan the development of large projects (strategy initiatives) for the following five years. Suppose there are four possible projects $A_{i}(i=1,2,3,4)$ to be evaluated. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the balanced scorecard methodology (Kaplan and Norton 1996) (it should be noted that all of them are of the benefit type): (1) $x_{1}$ : Financial perspective; (2) $x_{2}$ : The customer satisfaction; (3) $x_{3}$ : Internal business process perspective; (4) $x_{4}$ : Learning and growth perspective. Suppose that the weight vector of the attributes is $w=(0.2,0.3,0.15,0.35)^{\mathrm{T}}$. In order to get more reasonable results, a decision organization is required to provide all the possible values that the alternative $A_{i}$ satisfies the attribute $x_{j}$ represented by a HFE $h_{i j}$. All the HFEs $h_{i j}(i, j=1,2,3,4)$ are contained in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 4}$ (see Table 1.16 (Xia and Xu 2011b)).

Table 1.16. The hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 4}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4,0.7\}$ | $\{0.2,0.6,0.8\}$ | $\{0.2,0.3,0.6,0.7,0.9\}$ | $\{0.3,0.4,0.5,0.7,0.8\}$ |
| $A_{2}$ | $\{0.2,0.4,0.7,0.9\}$ | $\{0.1,0.2,0.4,0.5\}$ | $\{0.3,0.4,0.6,0.9\}$ | $\{0.5,0.6,0.8,0.9\}$ |
| $A_{3}$ | $\{0.3,0.5,0.6,0.7\}$ | $\{0.2,0.4,0.5,0.6\}$ | $\{0.3,0.5,0.7,0.8\}$ | $\{0.2,0.5,0.6,0.7\}$ |
| $A_{4}$ | $\{0.3,0.5,0.6\}$ | $\{0.2,0.4\}$ | $\{0.5,0.6,0.7\}$ | $\{0.8,0.9\}$ |

To get the ranking of the alternatives, the proposed method is used as follows:
Step 1. Utilize the GHFWA operator to obtain the HFEs $h_{i}(i=1,2,3,4)$ for the projects $A_{i}(i=1,2,3,4)$. Take the project $A_{4}$ for an example, and let $\dot{\tau}(t)=\log \left(\frac{\zeta+(1-\zeta) t}{t}\right), \lambda=1$ and $\zeta=2$, then we have

$$
\begin{aligned}
h_{4} & =\text { ATS-HFWA }(\{0.3,0.5,0.6\},\{0.2,0.4\},\{0.5,0.6,0.7\},\{0.8,0.9\}) \\
& =\{0.5296,0.5450,0.5631,0.5633,0.5756,0.5778,0.5826,0.5899,0.5949
\end{aligned}
$$

$$
0.5967,0.6067,0.6068,0.6132,0.6172,0.6203,0.6247,0.6303,0.6361
$$

$$
0.6377,0.6458,0.6460,0.6529,0.6565,0.6584,0.6624,0.6686,0.6729
$$

$$
0.6744,0.6828,0.6830,0.6884,0.6943,0.6980,0.7076,0.7089,0.7217\}
$$

As the values of the parameters $\lambda$ and $\zeta$ change, we can get different results for alternatives (see Table 1.17).

Step 2. Compute the scores $s\left(h_{i}\right)(i=1,2,3,4)$ of $h_{i}(i=1,2,3,4)$, which are shown in Table 1.17. By ranking $s\left(h_{i}\right)(i=1,2,3,4)$, we can get the priority of the alternatives $A_{i}(i=1,2,3,4)$ as the values of the parameters $\lambda$ and $\zeta$ change, which are listed in Table 1.17 (Xia and Xu 2011b).

Table 1.17. Scores obtained by the ATS-GHFWA operator and the rankings

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1, \zeta=2$ | 0.5487 | 0.5833 | 0.5091 | 0.6343 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| $\lambda=2, \zeta=2$ | 0.5793 | 0.6195 | 0.5332 | 0.6670 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| $\lambda=5, \zeta=2$ | 0.6504 | 0.6973 | 0.5897 | 0.7484 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| $\lambda=10, \zeta=2$ | 0.6942 | 0.7462 | 0.6291 | 0.7980 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |
| $\lambda=20, \zeta=2$ | 0.7215 | 0.7751 | 0.6558 | 0.8244 | $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$ |

From Table 1.17, we can find that the scores obtained by the ATS-GHFWA operator become bigger as the value of the parameter $\lambda$ increases, and the DMs can choose the values of $\lambda$ according to their preferences. In Step 2, if we use the ATS-GHFWG operator instead of the ATS-GHFWA operator to aggregate the values of the alternatives, the scores and the rankings of the alternatives are listed in Table 1.18 (Xia and Xu 2011b).

Table 1.18. Scores obtained by the ATS-GHFWG operator and the rankings

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1, \zeta=2$ | 0.4875 | 0.4817 | 0.4735 | 0.5332 | $A_{4} \succ A_{1} \succ A_{3} \succ A_{2}$ |
| $\lambda=2, \zeta=2$ | 0.4529 | 0.4343 | 0.4543 | 0.4769 | $A_{4} \succ A_{1} \succ A_{3} \succ A_{2}$ |
| $\lambda=5, \zeta=2$ | 0.3741 | 0.3510 | 0.3996 | 0.3844 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=10, \zeta=2$ | 0.3280 | 0.3036 | 0.3613 | 0.3383 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=20, \zeta=2$ | 0.3013 | 0.2758 | 0.3393 | 0.3119 | $A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |

It is pointed out that the ranking of the alternatives may change when the values of the parameters in the ATS-GHFWG operator change. By analyzing Tables 1.17 and 1.18, we can find that the scores obtained by the ATS-GHFWG operator become smaller as the values of the parameter $\lambda$ increase, but the values obtained by the ATS-GHFWA operator are always bigger than the ones obtained by the ATS-GHFWG operator.

### 1.7 Hesitant Multiplicative Aggregation

By giving a careful check, we can find that the HFE is described by using the 0.10.9 scale (see Table 1.19), which is uniformly distributed between 0 and 1 , but an asymmetric distribution, such as the 1-9 scale (Saaty 1980, see Table 1.19) may be more suitable in many practical problems, especially when we describe the priority degree that an alternative is prior to another in a decision making problem, it is because that our preferences are not symmetric distributed between "preferred" and "not preferred" but sometimes asymmetric to our intuitions. The law of diminishing marginal utility in economics is a good example. To invest the same resources to a company with bad performance and to a company with good performance, the former enhances more quickly than the latter. In other words, the gap between the grades expressing good information should be bigger than the one between the grades expressing bad information. Saaty's 1-9 scale is a useful tool to deal with such a situation. Motivated by this idea, we propose the hesitant multiplicative set (HMS) which describes the membership degree that an element to a set by using Saaty's 1-9 scale instead of the 0.1-0.9 scale in HFSs.

Table 1.19. The comparison between the 0.1-0.9 scale and the 1-9 scale

| $1-9$ scale | $0.1-0.9$ scale | Meaning |
| :---: | :---: | :---: |
| $\frac{1}{9}$ | 0.1 | Extremely not preferred |
| $\frac{1}{7}$ | 0.2 | Very strongly not preferred |
| $\frac{1}{5}$ | 0.3 | Strongly not preferred |
| $\frac{1}{3}$ | 0.4 | Moderately not preferred |
| 1 | 0.5 | Equally preferred |
| 3 | 0.6 | Moderately preferred |
| 5 | 0.7 | Strongly preferred |
| 7 | 0.8 | Very strongly preferred <br> 9 |
| Other values between 1 <br> and 9 | Other values between 0 <br> and 1 | Intermediate values used to <br> present compromise |

Given an example, if we provide the degree that the alternative $A_{i}$ is prior to the alternative $A_{j}$, a decision organization is authorized to get a more reasonable result, if some DMs in the decision organization provide $\frac{1}{7}$, some provide $\frac{1}{3}$, and others may provide 5 , and these three parts cannot persuade each other, therefore, the preference information that that the alternative $A_{i}$ is prior to the alternative $A_{j}$ can be represented by a $\operatorname{HME}\left\{\frac{1}{7}, \frac{1}{3}, 5\right\}$, which is the basic unit of a HMS.

It is noted that all t -norms and t -conorms are only suitable for the values between 0 and 1 , we can do some extensions for the usual $t$-norms and $t$ conorms and give the following definitions.

Definition 1.46 (Xia and Xu 2011c). A function $E T:(0,+\infty) \times(0,+\infty) \rightarrow(0,+\infty)$ is called an extended t-norm if it satisfies the following four conditions:
(1) $E T(+\infty, x)=x$, for all $x$.
(2) $E T(x, y)=E T(y, x)$, for all $x$ and $y$.
(3) $E T(x, E T(y, z))=E T(E T(x, y), z)$, for all $x, y$ and $z$.
(4) If $x \leq x^{\prime}$ and $y \leq y^{\prime}$, then $E T(x, y) \leq E T\left(x^{\prime}, y^{\prime}\right)$.

Definition 1.47 (Xia and Xu 2011c). A function $E S:(0,+\infty) \times(0,+\infty) \rightarrow(0,+\infty)$ is called an extended $t$-conorm if it satisfies the following four conditions:
(1) $E S(0, x)=x$, for all $x$.
(2) $E S(x, y)=E S(y, x)$, for all $x$ and $y$.
(3) $E S(x, E S(y, z))=E S(E S(x, y), z)$, for all $x, y$ and $z$.
(4) If $x \leq x^{\prime}$ and $y \leq y^{\prime}$, then $E S(x, x) \leq E S\left(x^{\prime}, x^{\prime}\right)$.

Definition 1.48 (Xia and Xu 2011c). An extended t-norm function $E T(x, y)$ is called an extended Archimedean t-norm if it is continuous and $E T(x, x)<x$ for
all $x \in(0,+\infty)$. An extended Archimedean t-norm is called strictly extended Archimedean t-norm if it is strictly increasing in each variable for $x, y \in(0,+\infty)$.

Definition 1.49 (Xia and Xu 2011c). A t-conorm function $E S(x, y)$ is called an extended Archimedean t-conorm if it is continuous and $E S(x, x)>x$ for all $x \in(0,+\infty)$. An extended Archimedean t-conorm is called strictly extended Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in(0,+\infty)$.

Similar to the Archimedean t-norm, the extended Archimedean t-norm can be expressed via its multiplicative generator $\dot{\tau}$ as $E T(x, y)=\dot{\tau}^{-1}(\dot{\tau}(x) \cdot \dot{\tau}(y))$, and similarly, applied to its dual extended t-conorm $\operatorname{ES}(x, y)=\dot{s}^{-1}(\dot{s}(x) \cdot \dot{s}(y))$ with $\dot{s}(t)=\dot{\tau}\left(\frac{1}{t}\right)$. We assume that a multiplicative generator of a continuous extended Archimedean t -norm is a strictly decreasing function $\dot{\tau}$ : $(0, \infty) \rightarrow(1, \infty)$ such that $\lim _{t \rightarrow+\infty} \dot{\tau}(t)=1$. If we assign specific forms to the function $\dot{\tau}$, then some specific extended t-norms and t-conorms can be obtained as follows (Xia and Xu 2011c):
(1) Let $\dot{\tau}(t)=\frac{1+t}{t}$, then $\dot{s}(t)=1+t, \dot{\tau}^{-1}(t)=\frac{1}{t-1}, \dot{s}^{-1}(t)=t-1$, and we have

$$
\begin{equation*}
E S^{A}(x, y)=(x+1)(y+1)-1, E T^{A}(x, y)=\frac{x y}{(x+1)(y+1)-x y} \tag{1.256}
\end{equation*}
$$

which we call an extended Algebraic t-conorm and extended Algebraic t-norm.
(2) Let $\dot{\tau}(t)=\frac{2+t}{t}$, then $\dot{s}(t)=2 t+1, \quad \dot{\tau}^{-1}(t)=\frac{2}{t-1}, \quad \dot{s}^{-1}(t)=\frac{t-1}{2}$, and we have

$$
\begin{equation*}
E S^{E}(x, y)=\frac{(2 x+1)(2 y+1)-1}{2}, E T^{E}(x, y)=\frac{2 x y}{(x+2)(y+2)-x y} \tag{1.257}
\end{equation*}
$$

which we call the extended Einstein t-conorm and extended Einstein t-norm.

$$
\begin{align*}
& \text { (3) Let } \quad \dot{\tau}(t)=\frac{\zeta+t}{t}, \quad \zeta>0, \quad \text { then } \quad \dot{s}(t)=\zeta t+1, \quad \dot{\tau}^{-1}(t)=\frac{\zeta}{t-1}, \\
& \dot{s}^{-1}(t)=\frac{t-1}{\zeta}, \text { and we have } \\
& E S_{\zeta}^{H}(x, y)=\frac{(\zeta x+1)(\zeta y+1)-1}{\zeta}, E T_{\zeta}^{H}(x, y)=\frac{\zeta x y}{(x+\zeta)(y+\zeta)-x y}, \zeta>0 \tag{1.258}
\end{align*}
$$

which we call an extended Hamacher t-conorm and extend Hamacher t-norm. Especially, if $\zeta=1$, then the extended Hamacher t-conorm and t-norm reduce to the extended Algebraic t-conorm and t-norm respectively; If $\zeta=2$, then the extended Hamacher t-conorm and t-norm reduce to the extended Einstein tconorm and t-norm respectively.

It is noted that the HFE uses the uniform distribution to express the membership degree of an element to a set, if we use the non-uniform distribution instead of uniform distribution scale to describe the membership degree that an element to a set, then we introduce the following definition:

Definition 1.50 (Xia and Xu 2011c). Let $X$ be a fixed set, a hesitant multiplicative set (HMS) on $X$ is in terms of a function $e$ that when applied to $X$ returns a subset of $\left[\frac{1}{a}, a\right], a>1$.

To be easily understood, we express the HMS by $\Upsilon=\{<x, e(x)>\mid x \in X\}$, where $e(x)$ is a set of some values in $\left[\frac{1}{a}, a\right]$, denoting the possible membership degrees of the element $x \in X$ to the set $\Upsilon$. For convenience, we call $e=e(x)$ a hesitant multiplicative element (HME) and $M$ the set of all HMEs. To rank the HMEs, we give the following comparison laws:

Definition 1.51 (Xia and Xu 2011c). For a HME $e, s(e)=\sqrt[1]{l_{e}} \sqrt{\prod_{\eta \in e}} \eta$ is called the score of $e$, where $l_{e}$ is the number of the elements in $e$. For two HMEs $e_{1}$ and $e_{2}$, if $s\left(e_{1}\right)>s\left(e_{2}\right)$, then $e_{1}>e_{2}$.

Based on the extended t-norms and t-norms, we introduce some operations for HMEs as follows:

Definition 1.52 (Xia and Xu 2011c). Let $e, e_{1}$ and $e_{2}$ be three HMEs, and $\lambda>0$, then
(1) $e^{\lambda}=\bigcup_{\eta \in e}\left\{\dot{\tau}^{-1}\left((\dot{\tau}(\eta))^{\lambda}\right)\right\}$.
(2) $\lambda e=\bigcup_{\eta \in e}\left\{\dot{s}^{-1}\left((\dot{s}(\eta))^{\lambda}\right)\right\}$.
(3) $e_{1} \otimes e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\eta_{1}\right) \cdot \dot{\tau}\left(\eta_{2}\right)\right)\right\}$.
(4) $e_{1} \oplus e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\eta_{1}\right) \cdot \dot{s}\left(\eta_{2}\right)\right)\right\}$.

The above functions $\dot{\tau}$ and $\dot{s}$ are defined as before.
Especially, if $\dot{\tau}(t)=\frac{1+t}{t}$, then we have
(5) $e^{\lambda}=\bigcup_{\eta \in e}\left\{\frac{\eta^{\lambda}}{(1+\eta)^{\lambda}-\eta^{\lambda}}\right\}, \lambda>0$.
(6) $\lambda e=\bigcup_{\eta \in e}\left\{(1+\eta)^{\lambda}-1\right\}, \lambda>0$.
(7) $e_{1} \otimes e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{\eta_{1} \eta_{2}}{\eta_{1}+\eta_{2}+1}\right\}$.
(8) $e_{1} \oplus e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\eta_{1}+\eta_{2}+\eta_{1} \eta_{2}\right\}$.

All the above are based on the extended Algebraic t-conorm and extended Algebraic t-norm.

If $\dot{\tau}(t)=\frac{2+t}{t}$, then we have
(9) $e^{\lambda}=\bigcup_{\eta \in e}\left\{\frac{2 \eta^{\lambda}}{(2+\eta)^{\lambda}-\eta^{\lambda}}\right\}, \lambda>0$.
(10) $\lambda e=\bigcup_{\eta \in e}\left\{\frac{(1+2 \eta)^{\lambda}-1}{2}\right\}, \lambda>0$.
(11) $e_{1} \otimes e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{2 \eta_{1} \eta_{2}}{\left(2+\eta_{1}\right)\left(2+\eta_{2}\right)-\eta_{1} \eta_{2}}\right\}$.
(12) $e_{1} \oplus e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{\left(1+2 \eta_{1}\right)\left(1+2 \eta_{2}\right)-1}{2}\right\}$.

All the above are based on the extended Einstein t-conorm and the extended Einstein t-norm.

If $\dot{\tau}(t)=\frac{\zeta+t}{t}, \zeta>0$, then we have
(13) $e^{\lambda}=\bigcup_{\eta \in e}\left\{\frac{\zeta \eta^{\lambda}}{(\eta+\zeta)^{\lambda}-\eta^{\lambda}}\right\}, \lambda>0$.
(14) $\lambda e=\bigcup_{\eta \in e}\left\{\frac{(\zeta \eta+1)^{\lambda}-1}{\zeta}\right\}, \lambda>0$.
(15) $e_{1} \otimes e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{\zeta \eta_{1} \eta_{2}}{\left(\eta_{1}+\zeta\right)\left(\eta_{2}+\zeta\right)-\eta_{1} \eta_{2}}\right\}$.
(16) $e_{1} \oplus e_{2}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{\left(\zeta \eta_{1}+1\right)\left(\zeta \eta_{2}+1\right)-1}{\zeta}\right\}$.

All the above are based on the extended Hamacher t-conorm and the extended Hamacher t-norm. Especially, if $\zeta=1$, then (13)-(16) reduce to (5)-(8); If $\zeta=2$, then (13)-(16) reduce to (9)-(12).

Some relationships can be further established for these operations on HMEs.

Theorem 1.48 (Xia and Xu 2011c). For three HMEs $e, e_{1}$ and $e_{2}$, $e^{c}=\bigcup_{\eta \in e}\left\{\frac{1}{\eta}\right\}$, the following are valid:
(1) $\left(e^{c}\right)^{\lambda}=(\lambda e)^{c}$.
(2) $\lambda\left(e^{c}\right)=\left(e^{\lambda}\right)^{c}$.
(3) $e_{1}^{c} \otimes e_{2}^{c}=\left(e_{1} \oplus e_{2}\right)^{c}$.
(4) $e_{1}^{c} \oplus e_{2}^{c}=\left(e_{1} \otimes e_{2}\right)^{c}$.

Proof. For three HMEs $e, e_{1}$ and $e_{2}$, we have
(1) $\left(e^{c}\right)^{\lambda}=\bigcup_{\eta \in e}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\frac{1}{\eta}\right)\right)^{\lambda}\right)\right\}=\left(\bigcup_{\eta \in e}\left\{\dot{s}^{-1}\left((\dot{s}(\eta))^{\lambda}\right)\right\}\right)^{c}=(\lambda e)^{c}$.
(2) $\lambda e^{c}=\bigcup_{\eta \in e}\left\{\dot{S}^{-1}\left(\left(\dot{s}\left(\frac{1}{\eta}\right)\right)^{\lambda}\right)\right\}=\left(\bigcup_{\eta \in e}\left\{\dot{\tau}^{-1}\left((\dot{\tau}(\eta))^{\lambda}\right)\right\}\right)^{c}=\left(e^{\lambda}\right)^{c}$.
(3) $e_{1}^{c} \oplus e_{2}^{c}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\frac{1}{\eta_{1}}\right) \cdot \dot{s}\left(\frac{1}{\eta_{2}}\right)\right)\right\}$
$=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{1}{\dot{\tau}^{-1}\left(\dot{\tau}\left(\eta_{1}\right) \cdot \dot{\tau}\left(\eta_{2}\right)\right)}\right\}=\left(e_{1} \otimes e_{2}\right)^{c}$.
(4) $e_{1}^{c} \otimes e_{2}^{c}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{\tau}^{-1}\left(\dot{\tau}\left(\frac{1}{\eta_{1}}\right) \cdot \dot{\tau}\left(\frac{1}{\eta_{2}}\right)\right)\right\}$
$=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\frac{1}{\dot{s}^{-1}\left(\dot{s}\left(\gamma_{1}\right) \cdot \dot{s}\left(\gamma_{2}\right)\right)}\right\}=\left(e_{1} \oplus e_{2}\right)^{c}$.
Next, we introduce several extended t-norm and t-conorm based aggregation operators for HMEs, and investigate their desirable properties:

Definition 1.53 (Xia and Xu 2011c). Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ the weight vector of them with $w_{i} \in[0,1]$, $i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$. If

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{HMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\oplus_{i=1}^{n} w_{i} e_{i} \tag{1.259}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{ATS}-H M W G\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\otimes_{i=1}^{n} e_{i}^{w_{i}} \tag{1.260}
\end{equation*}
$$

then ATS-HMWA and ATS-HFWG are called an extended Archimedean t-norm and $t$-conorm based hesitant multiplicative weighted averaging (ATS-HMWA) operator and an extended Archimedean t-norm and t-conorm based hesitant multiplicative weighted geometric (ATS-HMWG) operator, respectively.

Theorem 1.49 (Xia and Xu 2011c). Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ the weight vector of them with $w_{i} \in[0,1]$, $i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$, then
$\operatorname{ATS-HMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}^{\bigcup}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\}$
and

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{HMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\} \tag{1.262}
\end{equation*}
$$

Proof. By using the mathematical induction on $n$ : For $n=2$, we have

$$
\begin{align*}
& \operatorname{ATS}-\operatorname{HMWA}\left(e_{1}, e_{2}\right)=\stackrel{2}{\oplus} w_{i=1} e_{i}=w_{1} e_{1} \oplus w_{2} e_{2} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{1}\right)\right)^{w_{1}}\right)\right) \cdot \dot{s}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{2}\right)\right)^{w_{2}}\right)\right)\right)\right\}  \tag{1.263}\\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{1}\right)\right)^{w_{1}} \cdot\left(\dot{s}\left(\eta_{2}\right)\right)^{w_{2}}\right)\right\}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{2}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\}
\end{align*}
$$

Suppose the first equation holds for $n=k$, that is,
$\operatorname{ATS}-\operatorname{HMWA}\left(e_{1}, e_{2}, \cdots, e_{k}\right)=\oplus_{i=1}^{k} w_{i} e_{i}=w_{1} e_{1} \oplus w_{2} e_{2} \oplus \cdots \oplus w_{k} e_{k}$

$$
\begin{equation*}
=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{k} \in e_{k}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{k}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\} \tag{1.264}
\end{equation*}
$$

then


$$
\begin{align*}
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{k} \in e_{k}}\left\{\dot{\tau}^{-1}\left(\prod_{i=1}^{k}\left(\dot{\tau}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\} \oplus \bigcup_{\eta_{k+1} \in e_{k+1}}\left\{\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{k+1}\right)\right)^{w_{k+1}}\right)\right\} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{k} \in e_{k}, \eta_{k+1} \in e_{k+1}}\left\{\dot{s}^{-1}\left(\dot{s}\left(\dot{s}^{-1}\left(\prod_{i=1}^{k}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right) \cdot \dot{s}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{k+1}\right)\right)^{w_{k+1}}\right)\right)\right)\right\} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{k} \in e_{k}, \eta_{k+1} \in e_{k+1}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{k}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}} \cdot\left(\dot{s}\left(\eta_{k+1}\right)\right)^{w_{k+1}}\right)\right\} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{k} \in e_{k}, \eta_{k+1} \in e_{k+1}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{k+1}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)\right\} \tag{1.265}
\end{align*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.

Definition 1.54 (Xia and Xu 2011c). Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ the weight vector of them with $w_{i} \in[0,1]$, $i=1,2, \ldots, n, \sum_{i=1}^{n} w_{i}=1$ and $\lambda>0$. If

$$
\begin{equation*}
\operatorname{ATS}-G H M W A ~\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\left(\bigoplus_{i=1}^{n} w_{i} e_{i}^{\lambda}\right)^{\frac{1}{\lambda}} \tag{1.266}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { ATS-GHMWG }\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\frac{1}{\lambda} \stackrel{n}{\otimes} \lambda e_{i}^{w_{i}} \tag{1.267}
\end{equation*}
$$

then ATS-GHMWA and ATS-GHMWG are called the extended Archimedean t-norm and t-conorm based generalized hesitant fuzzy weighted averaging (ATSGHMWG) operator and the extended Archimedean t-norm and t-conorm based generalized hesitant fuzzy weighted geometric (ATS-GHFWG) operator, respectively. Especially, if $\lambda=1$, then the ATS-GHMWA and ATS-GHMWG operators reduce to the ATS-HMWA and ATS-HMWG operators, respectively.

Theorem 1.50 (Xia and Xu 2011c). Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ the weight vector of them with $w_{i} \in[0,1]$, $i=1,2, \ldots, n, \sum_{i=1}^{n} w_{i}=1$, and $\lambda>0$, then

ATS-GHMWA $\left(e_{1}, e_{2}, \cdots, e_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}^{\cup}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \tag{1.268}
\end{equation*}
$$

and
$\operatorname{ATS-GHMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)$

$$
\begin{equation*}
=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}^{\bigcup}\left\{\dot{s}^{-1}\left(\left(\dot{s}\left(\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \tag{1.269}
\end{equation*}
$$

Proof. Since $e_{i}^{\lambda}=\bigcup_{\eta_{i} \in e_{i}}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right\}$, then we have

$$
\begin{equation*}
\oplus_{i=1}^{n} w_{i} e_{i}^{\lambda}=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right\} \tag{1.270}
\end{equation*}
$$

and
$\operatorname{ATS-GHMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)=\left(\bigoplus_{i=1}^{n} w_{i} e_{i}^{\lambda}\right)^{\frac{1}{\lambda}}$

$$
\begin{equation*}
=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \tag{1.271}
\end{equation*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.

Some properties of the ATS-GHMWA and ATS-GHMWG operators can be discussed as follows:

Theorem 1.51 (Xia and Xu 2011c). If all $e_{i}(i=1,2, \ldots, n)$ are equal, i.e., $e_{i}=e$, for all $i$, then

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=e \tag{1.272}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHMWG}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=e \tag{1.273}
\end{equation*}
$$

Proof. Let $e_{i}=e$, we have
ATS-GHMWA $\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\operatorname{ATS}-G H M W A(e, e, \ldots, e)$

$$
\begin{align*}
& =\oplus_{i=1}^{n} w_{i} e=\bigcup_{\eta \in e}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left((\dot{\tau}(\eta))^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \\
& =\bigcup_{\eta \in e}\{\eta\}=e \tag{1.274}
\end{align*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.
Theorem 1.52 (Xia and Xu 2011c). Let $e_{i}$ and $e_{i}^{*}(i=1,2, \ldots, n)$ be two collections of HMEs, if $\eta_{i} \leq \eta_{i}^{*}$ for all $\eta_{i} \in e_{i}$ and $\eta_{i}^{*} \in e_{i}^{*}$, then

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right) \leq \operatorname{ATS}-G H M W A ~\left(e_{1}^{*}, e_{2}^{*}, \ldots, e_{n}^{*}\right) \tag{1.275}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHMWG}\left(e_{1}, e_{2}, \ldots, e_{n}\right) \leq \operatorname{ATS}-\operatorname{GHMWG}\left(e_{1}^{*}, e_{2}^{*}, \ldots, e_{n}^{*}\right) \tag{1.276}
\end{equation*}
$$

Proof. We have known that $\dot{s}(t)=\dot{\tau}\left(\frac{1}{t}\right)$, and $\dot{\tau}:(0,+\infty) \rightarrow(1,+\infty)$ is a strictly decreasing function, then $\dot{s}(t)$ is a strictly increasing function. Since $\eta_{i} \leq \eta_{i}^{*}$ for all $\eta_{i} \in e_{i}$ and $\eta_{i}^{*} \in e_{i}^{*}$, then we have

$$
\begin{align*}
& \dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right) \\
& \quad \leq \dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}^{*}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right) \tag{1.277}
\end{align*}
$$

then

$$
\begin{equation*}
\operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right) \leq \operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}^{*}, e_{2}^{*}, \ldots, e_{n}^{*}\right) \tag{1.278}
\end{equation*}
$$

Similarly, we can prove another part of the theorem, which completes the proof.
Theorem 1.53 (Xia and Xu 2011 c$)$. Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs, and $e^{+}=\bigcup_{\eta_{i} \in e_{i}} \max _{i}\left\{\eta_{i}\right\}, e^{-}=\bigcup_{\eta_{i} \in e_{i}} \min _{i}\left\{\eta_{i}\right\}$, then

$$
\begin{equation*}
e^{-} \leq \operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right) \leq e^{+} \tag{1.279}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{-} \leq \operatorname{ATS}-G H M W G\left(e_{1}, e_{2}, \ldots, e_{n}\right) \leq e^{+} \tag{1.280}
\end{equation*}
$$

If the multiplicative generator $\dot{\tau}$ is assigned different forms, then some specific aggregation operators can be obtained as follows:

Case 1. If $\dot{\tau}(t)=\frac{1+t}{t}$, then the ATS-GHMWA and ATS-GHMWG operators reduce to the following:

$$
\begin{equation*}
\operatorname{GHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{\left(\dot{\eta}^{+}\right)^{\frac{1}{\lambda}}-\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.281}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{GHMWG}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{\left(\ddot{\eta}^{+}\right)^{\frac{1}{\lambda}}-\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.282}
\end{equation*}
$$

where

$$
\begin{equation*}
\dot{\eta}^{+}=\prod_{i=1}^{n}\left(1+\eta_{i}\right)^{\lambda w_{i}}, \dot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(1+\eta_{i}\right)^{\lambda}-\eta_{i}^{\lambda}\right)^{w_{i}} \tag{1.283}
\end{equation*}
$$

and

$$
\begin{equation*}
\ddot{\eta}^{+}=\prod_{i=1}^{n}\left(1+\eta_{i}\right)^{\lambda w_{i}}, \ddot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(1+\eta_{i}\right)^{\lambda}-1\right)^{w_{i}} \tag{1.284}
\end{equation*}
$$

which are the generalized hesitant multiplicative weighted averaging (GHMWA) operator and the generalized hesitant multiplicative weighted geometric (GHMWG) operator, respectively.

Case 2. If $\dot{\tau}(t)=\frac{2+t}{t}$, then the ATS-GHMWA and ATS-GHMWG operators reduce to the following:

$$
\begin{equation*}
\operatorname{GEHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{2\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{\left(\dot{\eta}^{+}+3 \dot{\eta}^{-}\right)^{\frac{1}{\lambda}}-\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.285}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{GEHMWG}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{\left(\ddot{\eta}^{+}+3 \ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}-\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{2\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.286}
\end{equation*}
$$

where

$$
\begin{equation*}
\dot{\eta}^{+}=\prod_{i=1}^{n}\left(\left(2+\eta_{i}\right)^{\lambda}+3 \eta_{i}^{\lambda}\right)^{w_{i}}, \dot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(2+\eta_{i}\right)^{\lambda}-\eta_{i}^{\lambda}\right)^{w_{i}} \tag{1.287}
\end{equation*}
$$

and

$$
\begin{equation*}
\ddot{\eta}^{+}=\prod_{i=1}^{n}\left(3+\left(1+2 \eta_{i}\right)^{\lambda}\right)^{w_{i}}, \ddot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(1+2 \eta_{i}\right)^{\lambda}-1\right)^{w_{i}} \tag{1.288}
\end{equation*}
$$

which are the generalized Einstein hesitant multiplicative weighted averaging (GEHMWA) operator and the generalized Einstein hesitant multiplicaative weighted geometric (GEHMWG) operator, respectively.

Case 3. If $\dot{\tau}(t)=\frac{\zeta+t}{t}, \zeta>0$, then the ATS-GHMWA and ATS-GHMWG operators reduce to the following:

$$
\begin{equation*}
\operatorname{HHMWA}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{\zeta\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{\left(\dot{\eta}^{+}+\left(\zeta^{2}-1\right) \dot{\eta}^{-}\right)^{\frac{1}{\lambda}}-\left(\dot{\eta}^{+}-\dot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.289}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{HHMWG}\left(e_{1}, e_{2}, \ldots, e_{n}\right)=\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{\left(\ddot{\eta}^{+}+\left(\zeta^{2}-1\right) \ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}-\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}{\zeta\left(\ddot{\eta}^{+}-\ddot{\eta}^{-}\right)^{\frac{1}{\lambda}}}\right\} \tag{1.290}
\end{equation*}
$$

where

$$
\begin{equation*}
\dot{\eta}^{+}=\prod_{i=1}^{n}\left(\left(\eta_{i}+\zeta\right)^{\lambda}+\left(\zeta^{2}-1\right) \eta_{i}^{\lambda}\right)^{w_{i}}, \dot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(\eta_{i}+\zeta\right)^{\lambda}\right)^{w_{i}} \tag{1.291}
\end{equation*}
$$

and

$$
\begin{equation*}
\ddot{\eta}^{+}=\prod_{i=1}^{n}\left(\zeta^{2}+\left(\zeta \eta_{i}+1\right)^{\lambda}-1\right)^{w_{i}}, \ddot{\eta}^{-}=\prod_{i=1}^{n}\left(\left(\zeta \eta_{i}+1\right)^{\lambda}-1\right)^{w_{i}} \tag{1.292}
\end{equation*}
$$

which are the generalized Hammer hesitant multiplicative weighted averaging (GHHMWA) operator and the generalized Hammer hesitant multiplicative weighted geometric (GHHMWG) operator, respectively. Especially, if $\zeta=1$, then the GHHMWA operator reduces to the GHMWA operator and the GHHMWG operator reduces to the GHMWG operator; If $\zeta=2$, then the GHHMWA operator reduces to the GEHMWA operator and the GHHMWG operator reduces to the GEHMWG operator.

Example 1.26 (Xia and Xu 2011c). Let $h_{1}=\left\{\frac{1}{4}, \frac{3}{7}\right\}, \quad h_{2}=\left\{\frac{2}{3}\right\} \quad$ and $h_{3}=\left\{\frac{1}{9}, \frac{3}{7}, \frac{2}{3}\right\}$ be three HMEs, whose weight vector is $w=(0.2,0.3,0.5)^{\mathrm{T}}$, then we can use the ATS-GHIMWA or ATS-GHIMWG operator to aggregate them, without loss of generality, let $\dot{\tau}(t)=\frac{\zeta+t}{t}, \zeta=3$, and $\lambda=2$, then we have

$$
\dot{h}=\operatorname{ATS}-G H M W A\left(h_{1}, h_{2}, h_{3}\right)=\{0.4298,0.4534,0.4985,0.5175,0.6148,0.6290\}
$$

$$
s(\dot{h})=0.5186
$$

$\ddot{h}=\operatorname{ATS}-\operatorname{GHFWG}\left(h_{1}, h_{2}, h_{3}\right)=\{0.2069,0.2274,0.4201,0.4794,0.5082,0.5981\}$

$$
s(\ddot{h})=0.3772
$$

As the values of the parameters $\zeta$ and $\lambda$ between 0 and 1 , the scores obtained by using the ATS-GHMWA and ATS-GHMWG operators are given in Figs. 1.10-1.11 (Xia and Xu 2011c), respectively.


Fig. 1.10. Scores obtained by the ATS-GHMWA operator


Fig. 1.11. Scores obtained by the ATS-GHFWG operator

It is noted that the scores obtained by the ATS-GHMWA operator are increasing as the values of the parameter $\lambda$ increase, while the scores obtained by the ATS-GHMWG operator are quite the opposite. However, the scores obtained by the ATS-GHMWA operator are always bigger than the ones obtained by the ATS-GHMWG operator.

Finally, we discuss the relationships of the developed aggregation operators:
Theorem 1.54 (Xia and Xu 2011c). Let $e_{i}(i=1,2, \cdots, n)$ be a collection of HMEs with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ such that $w_{i} \in[0,1]$, $i=1,2, \ldots, n, \sum_{i=1}^{n} w_{i}=1$ and $\lambda>0$, then
(1) ATS-HMWA $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\left(\operatorname{ATS}-\operatorname{HMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c}$.
(2) ATS-HMWG $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\left(\operatorname{ATS}-\operatorname{HMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c}$.
(3) ATS-GHMWA $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\left(\operatorname{ATS-GHMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c}$.
(4) ATS-GHMWG $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\left(\operatorname{ATS-GHMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c}$.

Proof. (1) ATS-HMWA $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\stackrel{n}{\oplus}{ }_{i=1} w_{i} e_{i}^{c}$

$$
\begin{aligned}
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\frac{1}{\eta_{i}}\right)\right)^{w_{i}}\right)\right\} \\
& =\underset{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}{\bigcup}\left\{\frac{1}{\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\eta_{i}\right)\right)^{w_{i}}\right)}\right\}=\left(\begin{array}{l}
n \\
\otimes=1 \\
i=1
\end{array} e_{i}^{w_{i}}\right)^{c}
\end{aligned}
$$

$$
=\left(\operatorname{ATS}-\operatorname{HMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c}
$$

(2) ATS-HMWG $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\prod_{i=1}^{n}\left(e_{i}^{c}\right)^{w_{i}}$

$$
\begin{aligned}
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\frac{1}{\eta_{i}}\right)\right)^{w_{i}}\right)\right\} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{1}{\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\eta_{i}\right)\right)^{w_{i}}\right)}\right\}=\left(\bigoplus_{i=1}^{n} w_{i} e_{i}\right)^{c} \\
& =\left(\operatorname{ATS}-\operatorname{HMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c} .
\end{aligned}
$$

(3) ATS-GHMWA $\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\left(\oplus_{i=1}^{n} w_{i}\left(e_{i}^{c}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$

$$
\begin{aligned}
& =\underset{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}{\bigcup}\left\{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\frac{1}{\eta_{i}}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \\
& =\underset{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}{\cup}\left\{\frac{1}{\dot{s}^{-1}\left(\left(\dot{s}\left(\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)}\right\} \\
& =(\frac{1}{\lambda} \underbrace{n}_{i=1}\left(\lambda e_{i}\right)^{w_{i}})^{c}=\left(\operatorname{ATS-GHMWG}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c} .
\end{aligned}
$$

(4) $\operatorname{ATS}-G H M W G\left(e_{1}^{c}, e_{2}^{c}, \cdots, e_{n}^{c}\right)=\frac{1}{\lambda} \otimes_{i=1}^{n} \lambda\left(e_{i}^{c}\right)^{w_{i}}$

$$
\begin{aligned}
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\dot{S}^{-1}\left(\left(\dot{s}\left(\dot{\tau}^{-1}\left(\prod_{i=1}^{n}\left(\dot{\tau}\left(\dot{s}^{-1}\left(\left(\dot{s}\left(\frac{1}{\eta_{i}}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)\right\} \\
& =\bigcup_{\eta_{1} \in e_{1}, \eta_{2} \in e_{2}, \cdots, \eta_{n} \in e_{n}}\left\{\frac{1}{\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\dot{s}^{-1}\left(\prod_{i=1}^{n}\left(\dot{s}\left(\dot{\tau}^{-1}\left(\left(\dot{\tau}\left(\eta_{i}\right)\right)^{\lambda}\right)\right)\right)^{w_{i}}\right)\right)\right)^{\frac{1}{\lambda}}\right)}\right\} \\
& =\left(\left(\bigoplus_{i=1}^{n} w_{i} e_{i}^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{c}=\left(\operatorname{ATS}-\operatorname{GHMWA}\left(e_{1}, e_{2}, \cdots, e_{n}\right)\right)^{c} .
\end{aligned}
$$

## Chapter 2

## Distance, Similarity, Correlation, Entropy Measures and Clustering Algorithms for Hesitant Fuzzy Information

Distance and similarity measures are fundamentally important in a variety of scientific fields such as decision making, pattern recognition, machine learning and market prediction, lots of studies have been done about this issue on fuzzy sets (Turksen and Zhong 1988; Liu 1992; Bustince 2000; Candan et al. 2000). Among them, the most widely used distance measures for two fuzzy sets are the Hamming distance, the normalized Hamming distance, the Euclidean distance, and the normalized Euclidean distance (Diamond and Kloeden 1994; Kacprzyk 1997; Chaudhuri and Rosenfeld 1999). Later, a number of other extensions of the above distance measures have been developed for linguistic fuzzy sets (Xu 2005b) and intuitionistic fuzzy sets (Grzegorzewski 2004; Hung and Yang 2004, 2007; Li and Cheng 2002; Li et al. 2007; Liang and Shi 2003; Mitchell 2003; Szmidt and Kacprzyk 2000, 2001a; Wang and Xin 2005; Xu 2007b; Xu and Chen 2008a). For example, based on the Hamming distance, Xu (2005b) introduced the concepts of deviation degrees and similarity degrees between two linguistic values, and between two linguistic preference relations, respectively. Li and Cheng (2002) generalized Hamming distance and Euclidean distance by adding a parameter and gave a similarity formula for intuitionistic fuzzy sets only based on the membership degrees and the non-membership degrees. Some authors (Liang and Shi 2003; Mitchell 2003; Szmidt and Kacprzyk 2000, 2001a; Wang and Xin 2005) improved Li and Cheng's method (2002). Hung and Yang (2004) and Grzegorzewski (2004) suggested a lot of similarity measures for intuitionistic fuzzy sets and interval-valued fuzzy sets based on Hausdorff metric. Xu and Chen (2008a) gave a comprehensive overview of distance and similarity measures for intuitionistic fuzzy sets and developed several continuous distance and similarity measures for intuitionistic fuzzy sets. Wu and Mendel (2009) generalized Jaccard's similarity measure for T-2 fuzzy sets (Mendel 1999) and proposed a new similarity measure for interval T-2 fuzzy sets and compared it with the existing five methods (Gorzalczany 1987; Bustince 2000; Mitchell 2005; Wu and Mendel 2008; Zeng and Li 2006). Yang and Lin (2009) gave a few similarity and inclusion measures for T-2 fuzzy sets and combined them with Yang and Shih (2001)'s algorithms as a clustering method for type-2 fuzzy data.

Distance or similarity measures have attached a lot of attention in the last decades due to the fact that they can be applied to many areas such as pattern recognition (Li and Cheng 2002), clustering analysis (Yang and Lin 2009), approximate reasoning (Wang et al. 2002), image processing (Pal and King 1981), medical diagnosis (Szmidt and Kacprzyk 2001a) and decision making (Xu 2010b). A lot of distance and similarity measures have been developed for fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanassov 1986), linguistic fuzzy sets (Xu 2005), T-2 fuzzy sets (Dubois and Prade 1980; Miyamoto 2005), and fuzzy multisets (Yager 1986; Miyamoto 2000). Recently, Xu and Xia (2011b) originally developed a series of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures were proposed.

Correlation indicates how well two variables move together in a linear fashion. In other words, correlation reflects a linear relationship between two variables. It is an important measure in data analysis, in particular in decision making, predicting market behavior, medical diagnosis, pattern recognition, and other real world problems concerning environmental, political, legal, economic, financial, social, educational and artistic systems, etc. (Szmidt and Kacprzyk 2010). Hung and Wu (2001) used the concept of "expected value" to define the correlation coefficient of fuzzy numbers, which lies in $[-1,1]$. Hong (2006) considered the computational aspect of the $T_{w}$-based extension principle when the principle is applied to a correlation coefficient of L-R fuzzy numbers and gave the exact solution of a fuzzy correlation coefficient without programming or the aid of computer resources. In intuitionistic fuzzy environments, Gerstenkorn and Mańko (1991) defined the correlation and correlation coefficient of intuitionistic fuzzy sets (Atanassov 1986). Bustince and Burillo (1995) introduced the concepts of correlation and correlation coefficient of interval-valued intuitionistic fuzzy sets and studied their properties. They also introduced two decomposition theorems of the correlation of interval-valued intuitionistic fuzzy sets, one in terms of the correlation of interval-valued fuzzy sets and the entropy of intuitionistic fuzzy sets, and the other theorem in terms of the correlation of intuitionistic fuzzy sets. Hung (2001) and Mitchell (2004) derived the correlation coefficient of intuitionistic fuzzy sets from a statistical viewpoint by interpreting an intuitionistic fuzzy set as an ensemble of ordinary fuzzy sets. Hung and Wu (2002) proposed a method to calculate the correlation coefficient of intuitionistic fuzzy sets by means of "centroid". The formula tells us not only the strength of relationship between intuitionistic fuzzy sets, but also whether the considered intuitionistic fuzzy sets are positively or negatively related. Xu (2006) gave a detailed survey on association analysis of intuitionistic fuzzy sets, and pointed out that most existing methods deriving association coefficients cannot guarantee that the association coefficient of any two intuitionistic fuzzy sets equals one if and only if these two intuitionistic fuzzy sets are the same. Xu et al. (2008) utilized a set-theoretic approach to derive the association coefficients of intuitionistic fuzzy sets taking into account all the three terms (membership degree, non-membership degree, and hesitation margin) describing an intuitionstic fuzzy set. Szmidt and Kacprzyk (2010) discussed a
concept of correlation for data represented as intuitionistic fuzzy set adopting the concepts from statistics, proposed a formula for measuring the correlation coefficient (lying in $[-1,1]$ ) of intuitionistic fuzzy sets and showed the importance to take into account all three terms describing intuitionistic fuzzy sets. Recently, Xu and Xia (2011b) investigated the distance and correlation measures for HFEs, and then discussed their properties in detail. Chen et al. (2013a) derived some correlation coefficient formulas for HFSs and applied them to clustering analysis under hesitant fuzzy environments.

Entropy and cross-entropy are also important research topics in the fuzzy set theory, which have been widely used in practical applications, such as pattern recognition (Li and Cheng 2002), approximate reasoning (Wang et al. 2002), clustering analysis (Yang and Lin 2009), image processing (Pal and King 1981) and decision making (Ye 2010), etc. Entropy, first mentioned by Zadeh (1968), is a measure of fuzziness. Since its appearance, entropy has received great attentions. De Luca and Termini (1972) put forward some axioms to describe the fuzziness degree of a fuzzy set (Zadeh 1965), and proposed several entropy formulas based on Shannon's function. Kaufmann (1975) introduced an entropy formula for a fuzzy set by a metric distance between its membership degree function and the membership function of its nearest crisp set. Another method presented by Yager (1979) is to view the fuzziness degree of a fuzzy set in terms of a lack of distinction between the fuzzy set and its complement. Later on, other entropies for fuzzy sets have been given from different views (Bhandari and Pal 1993; Fan 2002; Kosko 1993; Liu 1992; Parkash et al. 2008; Shang and Jiang 1997). Since the concepts of interval-valued fuzzy set (Zadeh 1975), intuitionistic fuzzy set (Atanassov 1986) and rough set (Pawlak 1991) were introduced, the corresponding entropy theories have been investigated over the last decades. Burillo and Bustince (1996) presented an entropy measure on interval-valued fuzzy sets and intuitionstic fuzzy sets. Zeng and Li (2006) proposed a new concept of entropy for interval-valued fuzzy sets with a different view from Burillo and Bustince (1996). Zhang et al. (2009) introduced an axiomatic definition of entropy for an interval-valued fuzzy set based on distance measure which is consistent with the axiomatic definition of entropy of a fuzzy set introduced by De Luca and Termimi (1972) and Liu (1992). Szmidt and Kacprzyk (2001b) proposed a non-probabilistic entropy measure for intuitionstic fuzzy sets. Sen and Pal (2009) proposed classes of entropy measures based on rough set theory and its certain generalizations, and performed rigorous theoretical analysis to provide some properties which they satisfy. Cross-entropy measures are mainly used to measure the discrimination information. Up to now, a lot of research has been done about this issue (Buşoniu et al. 2011; Grzegorzewski 2004; Hung and Yang 2004, 2007, 2008; Li and Cheng 2002; Liang and Shi 2003; Mitchell 2003; Li et al. 2007; Li et al. 2009; Szmidt and Kacprzyk 2000; Xu and Chen 2008a). Vlachos and Sergiadis (2007) introduced the concepts of discrimination information and cross-entropy for intuitionistic fuzzy sets and revealed the connection between the notions of entropies for fuzzy sets and intuitionistic fuzzy sets in terms of fuzziness and intuitionism. Hung and Yang (2008) constructed
$J$-divergence of intuitionistic fuzzy sets and introduced some useful distance and similarity measures between two intuitionistic fuzzy sets, and applied them to clustering analysis and pattern recognition. Based on which, Xia and Xu (2012b) proposed some cross-entropy and entropy formulas for intuitionstic fuzzy sets and applied them to group decision making. The relationships among the entropy, cross-entropy and similarity measures have also attracted great attention (Liu 1992; Zeng and Guo 2008; Zeng and Li 2006; Zhang and Jiang 2010; Zhang et al. 2010). For example, Liu (1992) systematically gave the axiomatic definitions of entropy, distance measure and similarity measure of fuzzy sets and discussed their basic relations. Zeng and Li (2006) discussed the relationship between the similarity measure and the entropy of interval-valued fuzzy sets in detail and proved three theorems that the similarity measure and the entropy of interval-valued fuzzy sets can be transformed by each other based on their axiomatic definitions. For interval-valued intuitionistic fuzzy sets (Atanassov and Gargov 1989), Zhang and Jiang (2009) and Zhang et al. (2010) proposed the concepts of entropy and cross-entropy and discussed the connections among some important information measures. From the literature review above, we can find that little research has been done about hesitant fuzzy information, it is very necessary to develop some entropy measures under hesitant fuzzy environment. To do so, Xu and Xia (2012a) developed the concepts of entropy and cross-entropy for hesitant fuzzy information, and discussed their desirable properties. They analyzed the relationships among the proposed entropy, cross-entropy, and similarity measures.

Clustering refers to a process that combines a set of objects (alternatives, people, events, etc.) into clusters with respect to the characteristics of data, and the objects belonging to a cluster have a higher similarity than that of different clusters. As one of the widely-adopted key tools in handling data information, clustering analysis has been applied to the fields of pattern recognition (Bezdek 1998), data mining (Han and Kamber 2000), information retrieval (Mizutani et al. 2008; Miyamoto 2003), and other real world problems concerning social, medical, biological, climatic, and financial systems, etc. (Chaira 2011; Kumar et al. 2011; Nikas and Low 2011; Zhao and Zhang 2011; Zhao et al. 2011). In a real world, data used for clustering may be uncertain and fuzzy, to deal with various types of fuzzy data, a number of clustering algorithms corresponding to different fuzzy environments (Wu et al. 2011) have been proposed, e.g., intuitionistic fuzzy clustering algorithms (Wang et al. 2011; Xu et al. 2008, 2011, 2013; Xu 2009a; Xu and Wu 2010; Zhao et al. 2013) involving the correlation coefficient formulas for intuitionistic fuzzy sets (Xu et al. 2008) and type-2 fuzzy clustering algorithms (Hwang and Rhee 2007; Yang and Lin 2009). However, under the group decision making situations, the evaluation information provided by different DMs (experts) may have an obvious difference. These fuzzy clustering schemes mentioned above are unable to incorporate the differences in the opinions of different DMs, that is, they are unsuitable to do clustering under hesitant fuzzy environments. HFSs can be used to solve the issue, because they avoid performing data aggregation and can directly reflect the differences of the opinions of different DMs. Chen et al. (2013a) used the derived correlation coefficient formulas to calculate the degrees of correlation among HFSs aiming at clustering different objects. Zhang and Xu (2013) extended
the agglomerative hierarchical clustering algorithm to do clustering hesitant fuzzy information. Chen et al. (2014) investigated the clustering technique for HFSs based on the K-means clustering algorithm which takes the results of hierarchical clustering as the initial input. Zhang and Xu (2012) proposed a minimal spanning tree (MST) algorithm-based clustering technique to make clustering analysis of HFSs via some hesitant fuzzy distances. In the following sections, we shall give a detail survey of distance, similarity, correlation and entropy measures and clustering algorithms for hesitant fuzzy information.

### 2.1 Distance and Similarity Measures for HFSs

We first introduce the axioms for distance and similarity measures of HFSs:
Definition 2.1 (Xu and Xia 2011b). Let $A_{1}$ and $A_{2}$ be two HFSs on $X$, then the distance measure between $A_{1}$ and $A_{2}$ is defined as $d\left(A_{1}, A_{2}\right)$, which satisfies the following properties:
(1) $0 \leq d\left(A_{1}, A_{2}\right) \leq 1$.
(2) $d\left(A_{1}, A_{2}\right)=0$ if and only if $A_{1}=A_{2}$.
(3) $d\left(A_{1}, A_{2}\right)=d\left(A_{2}, A_{1}\right)$.

Definition 2.2 (Xu and Xia 2011b). Let $A_{1}$ and $A_{2}$ be two HFSs on $X$, then the similarity measure between $A_{1}$ and $A_{2}$ is defined as $\bar{s}\left(A_{1}, A_{2}\right)$, which satisfies the following properties:
(1) $0 \leq \bar{s}\left(A_{1}, A_{2}\right) \leq 1$.
(2) $\bar{s}\left(A_{1}, A_{2}\right)=1$ if and only if $A_{1}=A_{2}$.
(3) $\bar{s}\left(A_{1}, A_{2}\right)=\bar{s}\left(A_{2}, A_{1}\right)$.

By analyzing Definitions 2.1 and 2.2, it is noted that $\bar{s}\left(A_{1}, A_{2}\right)=1-d\left(A_{1}, A_{2}\right)$, accordingly, we mainly discuss the distance measures for HFSs in this section, and the corresponding similarity measures can be obtained easily.

In most cases, $l_{h_{h_{1}}\left(x_{i}\right)} \neq l_{h_{h_{2}}\left(x_{i}\right)}$, and for convenience, let $l_{x_{i}}=\max \left\{l_{h_{h_{1}}\left(x_{i}\right)}, l_{h_{h_{2}}\left(x_{i}\right)}\right\}$ for each $x_{i}$ in $X$. To operate correctly, we should extend the shorter one until both of them have the same length when we compare them. To extend the shorter one, the best way is to add a value in it. In fact, we can extend the shorter one by adding any value in it which mainly depends on the

DMs' risk preferences. The optimists anticipate desirable outcomes and may add the maximum value, while the pessimists expect unfavorable outcomes and may add the minimum value. For example, let $h_{A_{1}}\left(x_{i}\right)=\{0.1,0.2,0.3\}$, $h_{A_{2}}\left(x_{i}\right)=\{0.4,0.5\}$, and $l_{h_{A_{1}}\left(x_{i}\right)}>l_{h_{A_{2}}\left(x_{i}\right)}$. To operate correctly, we should extend $h_{A_{2}}\left(x_{i}\right)$ until it has the same length with $h_{A_{1}}\left(x_{i}\right)$, the optimist may extend $h_{A_{2}}\left(x_{i}\right)$ as $h_{A_{2}}\left(x_{i}\right)=\{0.4,0.5,0.5\}$ and the pessimist may extend it as $h_{A_{2}}\left(x_{i}\right)=\{0.4,0.4,0.5\}$. The results may be different if we extend the shorter one by adding different values, which is reasonable because the DMs' risk preferences can directly influence the final decision. The same situation can also be found in many existing references (Liu and Wang 2007; Merigó and Gil-Lafuente 2009; Merigó and Casanovas 2009). Here, we assume that the DMs are all pessimistic (with the same reason, the other situations can be studied similarly).

Drawing on the well-known Hamming distance and Euclidean distance, Xu and Xia (2011b) defined a hesitant normalized Hamming distance:

$$
\begin{equation*}
d_{1}\left(A_{1}, A_{2}\right)=\frac{1}{n} \sum_{i=1}^{n}\left[\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|\right] \tag{2.1}
\end{equation*}
$$

and a hesitant normalized Euclidean distance:

$$
\begin{equation*}
d_{2}\left(A_{1}, A_{2}\right)=\left[\frac{1}{n} \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right)\right]^{\frac{1}{2}} \tag{2.2}
\end{equation*}
$$

where $h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)$ and $h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)$ are the $j$ th largest values in $h_{A_{1}}\left(x_{i}\right)$ and $h_{A_{2}}\left(x_{i}\right)$, respectively, which will be used thereafter.

Xu and Xia (2011b) further extended the above distance measures and defined a generalized hesitant normalized distance:

$$
\begin{equation*}
d_{3}\left(A_{1}, A_{2}\right)=\left[\frac{1}{n} \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right]^{\frac{1}{\lambda}}\right. \tag{2.3}
\end{equation*}
$$

where $\lambda>0$.
It is noted that the parameter $\lambda$ provides the DMs more choices and can be assigned different values according to different DMs. It is motivated by the generalized idea provided by Yager (2004a), which has been widely applied to decision making (Beliakov 2005; Merigó and Gil-Lafuente 2009; Xu and Xia 2011c; Zhao et al. 2010; Zhou and Chen 2011).

Especially, if $\lambda=1$, then the generalized hesitant normal distance reduces to the hesitant normalized Hamming distance; If $\lambda=2$, then it reduces to the hesitant normalized Euclidean distance.

If we apply Hausdorff metric to the distance measure, then a generalized hesitant normalized Hausdorff distance is given as:

$$
\begin{equation*}
d_{4}\left(A_{1}, A_{2}\right)=\left[\frac{1}{n} \sum_{i=1}^{n} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right]^{\frac{1}{\lambda}} \tag{2.4}
\end{equation*}
$$

where $\lambda>0$.
Now we discuss two special cases of the generalized hesitant normalized Hausdorff distance (Xu and Xia 2011b):
(1) If $\lambda=1$, then $d_{4}\left(A_{1}, A_{2}\right)$ becomes a hesitant normalized HammingHausdorff distance:

$$
\begin{equation*}
d_{5}\left(A_{1}, A_{2}\right)=\frac{1}{n} \sum_{i=1}^{n} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right| \tag{2.5}
\end{equation*}
$$

(2) If $\lambda=2$, then $d_{4}\left(A_{1}, A_{2}\right)$ becomes a hesitant normalized EuclideanHausdorff distance:

$$
\begin{equation*}
d_{6}\left(A_{1}, A_{2}\right)=\left[\frac{1}{n} \sum_{i=1}^{n} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right]^{\frac{1}{2}} \tag{2.6}
\end{equation*}
$$

Combining the above equations, Xu and Xia (2011b) defined a hybrid hesitant normalized Hamming distance, a hybrid hesitant normalized Euclidean distance, and a generalized hybrid hesitant normalized distance as follows, respectively:

$$
\begin{align*}
& d_{7}\left(A_{1}, A_{2}\right)=\frac{1}{2 n} \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|\right)  \tag{2.7}\\
& d_{8}\left(A_{1}, A_{2}\right)=\left[\frac{1}{2 n} \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{x_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right)\right]^{\frac{1}{2}}  \tag{2.8}\\
& d_{9}\left(A_{1}, A_{2}\right)=\left[\frac{1}{2 n} \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right)\right]^{\frac{1}{2}} \tag{2.9}
\end{align*}
$$

where $\lambda>0$.

Usually, the weight of each element $x_{i} \in X$ should be taken into account, so Xu and Xia (2011b) presented the following weighted distance measures for HFSs:

Assume that the weight of the element $x_{i} \in X$ is $w_{i}(i=1,2, \cdots, n)$ with $w_{i} \in[0,1], i=1,2, \ldots, n$, and $\sum_{i=1}^{n} w_{i}=1$, then we get a generalized hesitant weighted distance:

$$
\begin{equation*}
d_{10}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}} \tag{2.10}
\end{equation*}
$$

and a generalized hesitant weighted Hausdorff distance:

$$
\begin{equation*}
d_{11}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right]^{\frac{1}{\lambda}} \tag{2.11}
\end{equation*}
$$

where $\lambda>0$.
In particular, if $\lambda=1$, then we obtain a hesitant weighted Hamming distance:

$$
\begin{equation*}
d_{12}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} w_{i}\left[\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|\right] \tag{2.12}
\end{equation*}
$$

and a hesitant weighted Hamming-Hausdorff distance:

$$
\begin{equation*}
d_{13}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} w_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right| \tag{2.13}
\end{equation*}
$$

If $\lambda=2$, then we get a hesitant weighted Euclidean distance:

$$
\begin{equation*}
d_{14}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right)\right]^{\frac{1}{2}} \tag{2.14}
\end{equation*}
$$

and a hesitant weighted Euclidean-Haudorff distance:

$$
\begin{equation*}
d_{15}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right]^{\frac{1}{2}} \tag{2.15}
\end{equation*}
$$

Furthermore, Xu and Xia (2011b) developed a generalized hybrid hesitant weighted distance combining the generalized hesitant weighted distance and the generalized hesitant weighted Hausdorff distance as:

$$
\begin{equation*}
d_{16}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}} \tag{2.16}
\end{equation*}
$$

where $\lambda>0$.
In the special cases where $\lambda=1,2, d_{16}\left(A_{1}, A_{2}\right)$ reduces to a hybrid hesitant weighted Hamming distance and a hybrid hesitant weighted Euclidean distance as follows, respectively:

$$
\begin{align*}
& d_{17}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|\right)  \tag{2.17}\\
& d_{18}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{i}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}+\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right|^{2}\right)\right]^{\frac{1}{2}} \tag{2.18}
\end{align*}
$$

We find that all the above distance measures are discrete, if both the universe of discourse and the weight of element are continuous, and the weight of $x \in X=[a, b]$ is $w(x)$, where $w(x) \in[0,1]$ and $\int_{a}^{b} w(x) d x=1$, then we define a continuous hesitant weighted Hamming distance, a continuous hesitant weighted Euclidean distance and a generalized continuous hesitant weighted distance as follows, respectively ( Xu and Xia 2011b):

$$
\begin{gather*}
d_{19}\left(A_{1}, A_{2}\right)=\int_{a}^{b} w(x)\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|\right) d x  \tag{2.19}\\
d_{20}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x)\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}\right) d x\right]^{\frac{1}{2}}  \tag{2.20}\\
d_{21}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x)\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda}\right) d x\right]^{\frac{1}{\lambda}} \tag{2.21}
\end{gather*}
$$

where $\lambda>0$.

If $w(x)=\frac{1}{b-a}$, for any $x \in[a, b]$, then the continuous hesitant weighted Hamming distance reduces to a continuous hesitant normalized Hamming distance:

$$
\begin{equation*}
d_{22}\left(A_{1}, A_{2}\right)=\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|\right) d x \tag{2.22}
\end{equation*}
$$

while the continuous hesitant weighted Euclidean distance reduces to a continuous hesitant normalized Euclidean distance:

$$
\begin{equation*}
d_{23}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}\right) d x\right]^{\frac{1}{2}} \tag{2.23}
\end{equation*}
$$

and the generalized continuous hesitant weighted distance reduces to a generalized continuous hesitant normalized distance:

$$
\begin{equation*}
d_{24}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda}\right) d x\right]^{\frac{1}{\lambda}} \tag{2.24}
\end{equation*}
$$

where $\lambda>0$.
Using the traditional Hausdorff metric, Xu and Xia (2011b) defined a generalized continuous hesitant weighted Hausdorff distance as

$$
\begin{equation*}
d_{25}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x) \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda} d x\right]^{\frac{1}{\lambda}} \tag{2.25}
\end{equation*}
$$

where $\lambda>0$.
In the special cases where $\lambda=1,2$, the generalized continuous hesitant weighted distance reduces to a continuous hesitant weighted Hamming-Hausdorff distance:

$$
\begin{equation*}
d_{26}\left(A_{1}, A_{2}\right)=\int_{a}^{b} w(x) \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right| d x \tag{2.26}
\end{equation*}
$$

and a continuous hesitant weighted Euclidean-Hausdorff distance:

$$
\begin{equation*}
d_{27}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x) \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2} d x\right]^{\frac{1}{2}} \tag{2.27}
\end{equation*}
$$

respectively.

If $w(x)=\frac{1}{b-a}$, for any $x \in[a, b]$, then the generalized continuous hesitant weighted distance becomes a generalized continuous hesitant normalized distance:

$$
\begin{equation*}
d_{28}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda} d x\right]^{\frac{1}{\lambda}} \tag{2.28}
\end{equation*}
$$

where $\lambda>0$, while the continuous hesitant weighted Euclidean-Hausdorff distance becomes a continuous hesitant normalized Hamming-Hausdorff distance:

$$
\begin{equation*}
d_{29}\left(A_{1}, A_{2}\right)=\frac{1}{b-a} \int_{a}^{b} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right| d x \tag{2.29}
\end{equation*}
$$

and the continuous hesitant weighted Euclidean-Hausdorff distance becomes a continuous hesitant normalized Euclidean-Hausdorff distance:

$$
\begin{equation*}
d_{30}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2} d x\right]^{\frac{1}{2}} \tag{2.30}
\end{equation*}
$$

Analogous to the generalized hybrid hesitant weighted distance, Xu and Xia (2011b) developed a generalized hybrid continuous hesitant weighted distance as:

$$
\begin{equation*}
d_{31}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x)\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda}+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}} \tag{2.31}
\end{equation*}
$$

where $\lambda>0$.
If $w(x)=\frac{1}{b-a}$, for any $x \in[a, b]$, then the generalized hybrid continuous hesitant weighted distance becomes a generalized hybrid continuous hesitant normalized distance:

$$
\begin{equation*}
d_{32}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}} \tag{2.32}
\end{equation*}
$$

where $\lambda>0$.
Let $\lambda=1,2$, we get a hybrid continuous hesitant weighted Hamming distance and a continuous hybrid continuous hesitant weighted Euclidean distance as:

$$
\begin{equation*}
d_{33}\left(A_{1}, A_{2}\right)=\int_{a}^{b} w(x)\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|\right) \tag{2.33}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{34}\left(A_{1}, A_{2}\right)=\left[\int_{a}^{b} w(x)\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}\right)\right]^{\frac{1}{2}} \tag{2.34}
\end{equation*}
$$

respectively.

$$
\text { Let } w(x)=\frac{1}{b-a} \text {, for any } x \in[a, b] \text {, then } d_{33}\left(A_{1}, A_{2}\right) \text { and } d_{34}\left(A_{1}, A_{2}\right)
$$

reduce to a hybrid continuous hesitant normalized Hamming distance:

$$
\begin{equation*}
d_{35}\left(A_{1}, A_{2}\right)=\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|\right) \tag{2.35}
\end{equation*}
$$

and a hybrid continuous hesitant normalized Euclidean distance:
$d_{36}\left(A_{1}, A_{2}\right)=\left[\frac{1}{b-a} \int_{a}^{b}\left(\frac{1}{2 l_{x}} \sum_{j=1}^{l_{x}}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}(x)-h_{A_{2}}^{\sigma(j)}(x)\right|^{2}\right)\right]^{\frac{1}{2}}$
respectively.
From the foregoing analysis, we find that the generalized hesitant weighted distance, the generalized hesitant weighted Hausdorff distance and the generalized hybrid hesitant weighted distance are three fundamental distance measures, based on which all the other developed distance measures can be obtained under some special conditions.

Xu and Xia (2011b) gave an example (adapted from Kahraman and Kaya (2010)) to illustrate the distance measures for HFSs:

Example 2.1 ( Xu and Xia 2011b). Energy is an indispensable factor for the social and economic development of societies. Thus the correct energy policy affects economic development and environment, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) $A_{i}(i=1,2,3,4,5)$ to be invested, and four attributes to be considered:
$x_{1}$ : Technological; $x_{2}$ : Environmental; $x_{3}$ : Socio-political; $x_{4}$ : Economic (more details about them can be found in Kahraman and Kaya (2010). The attribute weight vector is $w=(0.15,0.3,0.2,0.35)^{\mathrm{T}}$. Several DMs are invited to evaluate the performances of the five alternatives. For an alternative under an attribute, although all the DMs provide their evaluated values, some of these values may be repeated. However, a value repeated more times does not indicate that it has more importance than other values repeated less times. For example, the value repeated one time may be provided by a DM who is an expert at this area, and the value repeated twice may be provided by two DMs who are not familiar with this
area. In such cases, the value repeated one time may be more important than the one repeated twice. To get a more reasonable result, it is better that the DMs give their evaluations anonymously. We only collect all the possible values for an alternative under an attribute, and each value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once. The HFSs are just a tool to deal with such cases, and all possible evaluations for an alternative under the attributes can be considered as a HFS. The results evaluated by the DMs are contained in a hesitant fuzzy decision matrix, shown in Table 2.1 (Xu and Xia (2011b)).

Table 2.1. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5,0.4,0.3\}$ | $\{0.9,0.8,0.7,0.1\}$ | $\{0.5,0.4,0.2\}$ | $\{0.9,0.6,0.5,0.3\}$ |
| $A_{2}$ | $\{0.5,0.3\}$ | $\{0.9,0.7,0.6,0.5,0.2\}$ | $\{0.8,0.6,0.5,0.1\}$ | $\{0.7,0.3,0.4\}$ |
| $A_{3}$ | $\{0.7,0.6\}$ | $\{0.9,0.6\}$ | $\{0.7,0.5,0.3\}$ | $\{0.6,0.4\}$ |
| $A_{4}$ | $\{0.8,0.7,0.4,0.3\}$ | $\{0.7,0.4,0.2\}$ | $\{0.8,0.1\}$ | $\{0.9,0.8,0.6\}$ |
| $A_{5}$ | $\{0.9,0.7,0.6,0.3,0.1\}$ | $\{0.8,0.7,0.6,0.4\}$ | $\{0.9,0.8,0.7\}$ | $\{0.9,0.7,0.6,0.3\}$ |

Suppose that the ideal alternative is $A^{*}=\{1\}$ seen as a special HFS, we can calculate the distance between each alternative and the ideal alternative using our distance measures. The shorter the distance, the better the alternative.

If we use the generalized hesitant weighted distance, the generalized hesitant Hausdorff distance, and the generalized hybrid hesitant weighted distance to calculate the deviations between each alternative and the ideal alternative, then we get the rankings of the alternatives, which are listed in Tables 2.2-2.4 ( Xu and Xia 2011b), respectively, when some values of the parameter are given. We find that the rankings are different as the parameter $\lambda$ (which can be considered as the DMs' risk attitude) changes, consequently, the proposed distance measures can provide the DMs more choices as different values of the parameter are given according to the DMs' risk attitude.

Table 2.2. Results obtained by the generalized hesitant weighted distance

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | $A_{5}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1$ | 0.4799 | 0.5027 | 0.4025 | 0.4292 | 0.3558 | $A_{5} \succ A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=2$ | 0.5378 | 0.5451 | 0.4366 | 0.5052 | 0.4129 | $A_{5} \succ A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=6$ | 0.6599 | 0.6476 | 0.5156 | 0.6704 | 0.5699 | $A_{3} \succ A_{5} \succ A_{2} \succ A_{1} \succ A_{4}$ |
| $\lambda=10$ | 0.7213 | 0.7046 | 0.5607 | 0.7373 | 0.6537 | $A_{3} \succ A_{5} \succ A_{2} \succ A_{1} \succ A_{4}$ |

Table 2.3. Results obtained by the generalized hesitant weighted Hausdorff distance

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | $A_{5}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1$ | 0.7800 | 0.7700 | 0.5300 | 0.6650 | 0.6200 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=2$ | 0.7849 | 0.7740 | 0.5441 | 0.6953 | 0.6473 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=6$ | 0.8043 | 0.7904 | 0.5889 | 0.7673 | 0.7163 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=10$ | 0.8216 | 0.8063 | 0.6156 | 0.7991 | 0.7597 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |

Table 2.4. Results obtained by the generalized hybrid hesitant weighted distance

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | $A_{5}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1$ | 0.6300 | 0.6363 | 0.4662 | 0.5471 | 0.4879 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=2$ | 0.6613 | 0.6595 | 0.4903 | 0.6002 | 0.5301 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=6$ | 0.7321 | 0.7190 | 0.5523 | 0.7188 | 0.6431 | $A_{3} \succ A_{5} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=10$ | 0.7628 | 0.7475 | 0.5748 | 0.7523 | 0.6850 | $A_{3} \succ A_{5} \succ A_{2} \succ A_{2} \succ A_{1}$ |

Xu and Chen (2008a) defined several ordered weighted distance measures whose prominent characteristic is that they can alleviate (or intensify) the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. This desirable characteristic makes the ordered weighted distance measures very useful in many actual fields such as group decision making, medical diagnosis, data mining, and pattern recognition. Yager (2010) generalized Xu and Chen (2008a)'s distance measures and provided a variety of ordered weighted averaging norms, based on which he proposed several similarity measures. Merigó and Gil-Lafuente (2010) introduced an ordered weighted averaging distance operator and gave its application in the selection of financial products. In what follows, we introduce some ordered distance measures for HFSs .

Motivated by the ordered weighted idea (Yager 1988), Xu and Xia (2011b) defined a hesitant ordered weighted Hamming distance:

$$
\begin{equation*}
d_{37}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} \omega_{i}\left[\frac{1}{l_{x_{\bar{\sigma}(i)}}} \sum_{j=1}^{l_{x \sigma(i)}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)\right|\right] \tag{2.37}
\end{equation*}
$$

and a hesitant ordered weighted Euclidean distance:

$$
\begin{equation*}
d_{38}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} \omega_{i}\left(\frac{1}{l_{x_{\bar{\sigma}(i)}}} \sum_{j=1}^{l_{x_{\bar{\sigma}(i)}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)\right|^{2}\right]^{\frac{1}{2}}\right. \tag{2.38}
\end{equation*}
$$

respectively, where $\sigma(j)$ is given as in Section 1.2 , and $\bar{\sigma}:(1,2, \cdots, n) \rightarrow$ $(1,2, \cdots, n)$ is a permutation satisfying

$$
\begin{gather*}
\frac{1}{l_{x_{\sigma(i+1)}}} \sum_{j=1}^{l_{x_{\sigma(i+1)}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\sigma(i+1)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\sigma(i+1)}\right)\right| \geq \frac{1}{l_{x_{\sigma(i)}}} \sum_{j=1}^{l_{x_{\sigma(i)}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\sigma(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\sigma(i)}\right)\right|, \\
i=1,2, \cdots, n-1 \tag{2.39}
\end{gather*}
$$

Generalizing $d_{37}\left(A_{1}, A_{2}\right)$ and $d_{38}\left(A_{1}, A_{2}\right), \mathrm{Xu}$ and Xia (2011b) defined a generalized hesitant ordered weighted distance measure:

$$
\begin{equation*}
d_{39}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} \omega_{i}\left(\frac{1}{l_{x_{\bar{\sigma}(i)}}} \sum_{j=1}^{l_{x_{\bar{\sigma}(i)}}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\bar{\sigma}(i)}\right)\right|^{\lambda}\right]^{\frac{1}{\lambda}}\right. \tag{2.40}
\end{equation*}
$$

where $\lambda>0$.
With the Hausdorff metric, Xu and Xia (2011b) developed a generalized hesitant ordered weighted Hausdorff distance as:

$$
\begin{equation*}
d_{40}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} \omega_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)\right|^{\lambda}\right]^{\frac{1}{\lambda}} \tag{2.41}
\end{equation*}
$$

where $\lambda>0$ and $\dot{\sigma}:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ is a permutation satisfying

$$
\begin{gather*}
\max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\sigma(i+1)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\sigma(i+1)}\right)\right| \geq \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\sigma(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\sigma(i)}\right)\right|, \\
i=1,2, \cdots, n-1 \tag{2.42}
\end{gather*}
$$

In what follows, we discuss two special cases of the generalized hesitant ordered weighted Hausdorff distance (Xu and Xia 2011b):
(1) If $\lambda=1$, then $d_{40}\left(A_{1}, A_{2}\right)$ reduces to a hesitant ordered weighted Hamming-Hausdorff distance:

$$
\begin{equation*}
d_{41}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} \omega_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)\right| \tag{2.43}
\end{equation*}
$$

(2) If $\lambda=2$, then $d_{40}\left(A_{1}, A_{2}\right)$ reduces to a hesitant ordered weighted Euclidean-Hausdorff distance:

$$
\begin{equation*}
d_{42}\left(A_{1}, A_{2}\right)=\left[\sum_{i=1}^{n} \omega_{i} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\dot{\sigma}(i)}\right)\right|^{2}\right]^{\frac{1}{2}} \tag{2.44}
\end{equation*}
$$

Combining $d_{39}\left(A_{1}, A_{2}\right)$ and $d_{40}\left(A_{1}, A_{2}\right), \mathrm{Xu}$ and Xia (2011b) developed a generalized hybrid hesitant ordered weighted distance as:

$$
\begin{align*}
& d_{43}\left(A_{1}, A_{2}\right) \\
= & {\left[\sum_{i=1}^{n} \omega_{i}\left(\frac{1}{2 l_{x_{\sigma(i)}}} \sum_{j=1}^{l_{\dot{\partial}(i)}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)\right|^{\lambda}+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\ddot{\partial}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\tilde{\sigma}(i)}\right)\right|^{\lambda}\right)\right]^{1 / \lambda} } \tag{2.45}
\end{align*}
$$

where $\lambda>0, \ddot{\sigma}:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ is a permutation such that

$$
\frac{1}{2 l_{x_{\tilde{\sigma}(i+1)}}} \sum_{j=1}^{l_{x(i+1)}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\tilde{\sigma}(i+1)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\tilde{\sigma}(i+1)}\right)\right|+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i+1)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\tilde{\sigma}(i+1)}\right)\right|
$$

$$
\begin{gather*}
\geq \frac{1}{2 l_{x_{\ddot{\sigma}(i)}}} \sum_{j=1}^{l_{x \ddot{x}(i)}}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)\right|+\frac{1}{2} \max _{j}\left|h_{A_{1}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)-h_{A_{2}}^{\sigma(j)}\left(x_{\ddot{\sigma}(i)}\right)\right| \\
i=1,2, \cdots, n-1 \tag{2.46}
\end{gather*}
$$

As the parameter and the weight vector change, some special cases can be obtained just as discussed before. Let $d_{o}$ denote the ordered distance measures defined above, then the ordered similarity measures for HFSs can be given as $s_{o}=1-d_{o}$.
Another important issue is the determination of the weight vectors associated with the ordered weighted distance measures. Inspired by Xu and Chen (2008a), below we give three ways to determine the weight vectors (Xu and Xia 2011b):

Considering each element in $A_{1}$ and $A_{2}$ as a special HFS, $d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)(i=1,2, \cdots, n)$ as given in this section, and denoting $\bar{\sigma}, \dot{\sigma}$ and $\ddot{\sigma}$ as $\sigma$, we have
(1) Let

$$
\begin{equation*}
\omega_{i}=\frac{d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)}{\sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right)}, i=1,2, \cdots, n \tag{2.47}
\end{equation*}
$$

then $\omega_{i+1} \geq \omega_{i} \geq 0, i=1,2, \cdots, n-1$, and $\sum_{i=1}^{n} \omega_{i}=1$.
(2) Let

$$
\begin{equation*}
\omega_{i}=\frac{e^{-d\left(h_{h_{1}}\left(x_{\sigma(i)}\right), h_{1_{2}}\left(x_{\sigma(i)}\right)\right)}}{\sum_{k=1}^{n} e^{-d\left(h_{h_{1}}\left(x_{\sigma(k)}\right), h_{h_{2}}\left(x_{\sigma(k)}\right)\right)}}, \quad i=1,2, \cdots, n \tag{2.48}
\end{equation*}
$$

then $0 \leq \omega_{i+1} \leq \omega_{i}, \quad i=1,2, \cdots, n-1$, and $\sum_{i=1}^{n} \omega_{i}=1$.
(3) Let

$$
\begin{equation*}
\dot{d}\left(h_{A_{1}}, h_{A_{2}}\right)=\frac{1}{n} \sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right) \tag{2.49}
\end{equation*}
$$

and

$$
\begin{align*}
& \ddot{d}\left(d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right), \dot{d}\left(h_{A_{1}}, h_{A_{2}}\right)\right) \\
& \quad=\left|d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)-\frac{1}{n} \sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right)\right| \tag{2.50}
\end{align*}
$$

then we define

$$
\begin{align*}
& \omega_{i}=\frac{1-\ddot{d}\left(\dot{d}\left(h_{A_{1}}, h_{A_{2}}\right), d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)\right)}{\sum_{k=1}^{n}\left(1-\ddot{d}\left(\dot{d}\left(h_{A_{1}}, h_{A_{2}}\right), d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)\right)\right)} \\
& =\frac{1-\left|d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)-\frac{1}{n} \sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right)\right|}{\sum_{k=1}^{n}\left(1-\left|d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)-\frac{1}{n} \sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right)\right|\right)}, \\
& i=1,2, \cdots, n \tag{2.51}
\end{align*}
$$

from which we get $\omega_{i} \geq 0, i=1,2, \cdots, n$, and $\sum_{i=1}^{n} \omega_{i}=1$.
We find that the weight vector derived from the formula in (1) is a monotonically decreasing sequence, the weight vector derived from the formula in (2) is a monotonically increasing sequence, and the weight vector derived from the formula in (3) combines the above two cases, i.e., the closer the value $d\left(h_{A_{1}}\left(x_{\sigma(i)}\right), h_{A_{2}}\left(x_{\sigma(i)}\right)\right)$ to the mean $\frac{1}{n} \sum_{k=1}^{n} d\left(h_{A_{1}}\left(x_{\sigma(k)}\right), h_{A_{2}}\left(x_{\sigma(k)}\right)\right)$, the more the weight $\omega_{i}$.

In Example 2.1, if the attribute weight vector is unknown, then we can use the ordered weighted distance measures to calculate the distance between each alternative and the ideal alternative. Without loss of generality, suppose that $d=d_{1}$ in the formula in (1), we use the generalized hesitant ordered weighted distance measure $d_{39}\left(A_{1}, A_{2}\right)$ to calculate the distance between each alternative and the ideal alternative. The derived results are shown in Table 2.5 ( Xu and Xia 2011b) with the different values of the parameter $\lambda$.

Table 2.5. Results obtained by the generalized hesitant ordered weighted distance

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | $A_{5}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda=1$ | 0.5085 | 0.5132 | 0.4014 | 0.4565 | 0.3600 | $A_{5} \succ A_{3} \succ A_{4} \succ A_{1} \succ A_{2}$ |
| $\lambda=2$ | 0.5584 | 0.5545 | 0.4327 | 0.5304 | 0.4222 | $A_{5} \succ A_{3} \succ A_{4} \succ A_{2} \succ A_{1}$ |
| $\lambda=6$ | 0.6604 | 0.6561 | 0.5149 | 0.6863 | 0.5915 | $A_{3} \succ A_{5} \succ A_{2} \succ A_{1} \succ A_{4}$ |
| $\lambda=10$ | 0.7160 | 0.7140 | 0.5639 | 0.7492 | 0.6771 | $A_{3} \succ A_{5} \succ A_{2} \succ A_{1} \succ A_{4}$ |

### 2.2 Distance and Correlation Measures for HFEs

Consider that the number of values in different HFEs maybe different, let $l_{h}$ be the number of values in a HFE $h$. We can also find that the values in the HFE are out of order, we can arrange them in any order. For the HFE $h$, let $\sigma:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ be a permutation satisfying $h_{\sigma_{(i)}} \leq h_{\sigma_{(i+1)}}$, $i=1,2, \cdots, l_{h}-1, \quad \rho:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ be a permutation satisfying $h_{\rho(i)} \geq h_{\rho(i+1)}, i=1,2, \cdots, l_{h}-1$ and $\varsigma:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ be any permutation of the values in $h$. To operator correctly, we suppose that two HFEs $h_{1}$ and $h_{2}$ have the same length $l$, when we compare them, and $h_{1}=h_{2}$ if and only if $h_{1}^{\sigma(i)}=h_{2}^{\sigma(i)}$, for $i=1,2, \cdots, l$.

Definition 2.3 (Xu and Xia 2011c). For two HFEs $h_{1}$ and $h_{2}$, the distance between $h_{1}$ and $h_{2}$, denoted as $d\left(h_{1}, h_{2}\right)$, should satisfy the following properties:
(1) $0 \leq d\left(h_{1}, h_{2}\right) \leq 1$.
(2) $d\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}=h_{2}$.
(3) $d\left(h_{1}, h_{2}\right)=d\left(h_{2}, h_{1}\right)$.

Based on Definition 2.3, we can give the following distance measures for HFEs:
(1) $d_{1}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|$.
(2) $d_{2}\left(h_{1}, h_{2}\right)=\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}}$.
(3) $d_{3}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}$.
(4) $d_{4}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}\right\}$.
(5) $d_{5}\left(h_{1}, h_{2}\right)=\frac{1}{2}\left(\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}\right)$.
(6) $d_{6}\left(h_{1}, h_{2}\right)=\frac{1}{2}\left(\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}}+\max _{i}\left\{h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right\}\right)$.

These distance measures are the extensions of the well-known distance measures (such as Hamming distance, Euclidean distance, and Hausdorff metric) for HFEs, and they have their own features which are discussed below:

Lemma 2.1 ( Xu and Xia 2011c). Let $0 \leq a_{1} \leq b_{1} \leq 1$ and $0 \leq a_{2} \leq b_{2} \leq 1$, then we have

$$
\begin{equation*}
\left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right| \leq\left|a_{1}-b_{2}\right|+\left|a_{2}-b_{1}\right| \tag{2.52}
\end{equation*}
$$

Proof. We can distinguish six cases, according to the signs of $a_{1}-a_{2}$, $b_{1}-b_{2}, a_{1}-b_{2}$ and $a_{2}-b_{1}$ :

Case 1. $a_{1} \leq b_{1} \leq a_{2} \leq b_{2}$. In this case, we have

$$
\begin{align*}
& \left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right| \\
& \quad=a_{2}-a_{1}+b_{2}-b_{1}-\left(b_{2}-a_{1}\right)-\left(a_{2}-b_{1}\right)=0 \tag{2.53}
\end{align*}
$$

Case 2. $a_{1} \leq a_{2} \leq b_{1} \leq b_{2}$. In this case, we have

$$
\begin{align*}
\mid a_{1} & -a_{2}\left|+\left|b_{1}-b_{2}\right|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right|\right. \\
& =a_{2}-a_{1}+b_{2}-b_{1}-\left(b_{2}-a_{1}\right)-\left(b_{1}-a_{2}\right)=2\left(a_{2}-b_{1}\right) \leq 0 \tag{2.54}
\end{align*}
$$

Case 3. $a_{2} \leq a_{1} \leq b_{1} \leq b_{2}$. In this case, we have

$$
\begin{align*}
& \left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right| \\
& \quad=a_{1}-a_{2}+b_{2}-b_{1}-\left(b_{2}-a_{1}\right)-\left(b_{1}-a_{2}\right)=2\left(a_{1}-b_{1}\right) \leq 0 \tag{2.55}
\end{align*}
$$

Case 4. $\quad a_{2} \leq a_{1} \leq b_{2} \leq b_{1}$. In this case, we have

$$
\begin{align*}
& \left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right| \\
& \quad=a_{1}-a_{2}+b_{1}-b_{2}-\left(b_{2}-a_{1}\right)-\left(b_{1}-a_{2}\right)=2\left(a_{1}-b_{2}\right) \leq 0 \tag{2.56}
\end{align*}
$$

Case 5. $a_{2} \leq b_{2} \leq a_{1} \leq b_{1}$. In this case, we have

$$
\begin{align*}
\left|a_{1}-a_{2}\right|+\mid b_{1} & -b_{2}\left|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right|\right. \\
& =a_{1}-a_{2}+b_{1}-b_{2}-\left(a_{1}-b_{2}\right)-\left(b_{1}-a_{2}\right)=0 \tag{2.57}
\end{align*}
$$

Case 6. $a_{1} \leq a_{2} \leq b_{2} \leq b_{1}$. In this case, we have

$$
\begin{align*}
& \left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right|-\left|a_{1}-b_{2}\right|-\left|a_{2}-b_{1}\right| \\
& \quad=a_{2}-a_{1}+b_{1}-b_{2}-\left(b_{2}-a_{1}\right)-\left(b_{1}-a_{2}\right)=2\left(a_{2}-b_{2}\right) \leq 0 \tag{2.58}
\end{align*}
$$

Lemma 2.2 ( Xu and Xia 2011c). Let $a_{1} \leq b_{1}$ and $a_{2} \leq b_{2}$, then $a_{1} a_{2}+b_{1} b_{2} \geq a_{1} b_{2}+a_{2} b_{1}$.

Proof. Since $a_{1} \leq b_{1}$ and $a_{2} \leq b_{2}$, then

$$
\begin{equation*}
a_{1} a_{2}+b_{1} b_{2}-a_{1} b_{2}-a_{2} b_{1}=a_{1}\left(a_{2}-b_{2}\right)+b_{1}\left(b_{2}-a_{2}\right)=\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right) \geq 0 \tag{2.59}
\end{equation*}
$$

which completes the proof.
Lemma 2.3 (Xu and Xia 2011c). Let $a_{1} \leq b_{1}$ and $a_{2} \leq b_{2}$, then

$$
\begin{equation*}
\left|a_{1}-a_{2}\right|^{2}+\left|b_{1}-b_{2}\right|^{2} \leq\left|a_{1}-b_{2}\right|^{2}+\left|a_{2}-b_{1}\right|^{2} \tag{2.60}
\end{equation*}
$$

Proof. From Lemma 2.2, it follows that

$$
\begin{aligned}
& \left|a_{1}-a_{2}\right|^{2}+\left|b_{1}-b_{2}\right|^{2}-\left|a_{1}-b_{2}\right|^{2}-\left|a_{2}-b_{1}\right|^{2} \\
& =\left(a_{1}\right)^{2}-2 a_{1} a_{2}+\left(a_{2}\right)^{2}+\left(b_{1}\right)^{2}-2 b_{1} b_{2}+\left(b_{2}\right)^{2}-\left(\left(a_{1}\right)^{2}-2 a_{1} b_{2}+\left(b_{2}\right)^{2}\right)
\end{aligned}
$$

$$
\begin{align*}
-\left(\left(a_{2}\right)^{2}-\right. & \left.2 a_{2} b_{1}+\left(b_{1}\right)^{2}\right) \\
& =-2 a_{1} a_{2}-2 b_{1} b_{2}+2 a_{1} b_{2}+2 a_{2} b_{1} \leq 0 \tag{2.61}
\end{align*}
$$

which completes the proof of the lemma.

Theorem 2.1 (Xu and Xia 2011c). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $d_{1}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right| \leq \frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\varsigma(i)}-h_{2}^{\zeta(i)}\right| \leq \frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|$.
(2) $d_{2}\left(h_{1}, h_{2}\right)=\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}} \leq \sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|^{2}}=\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|^{2}}$.
(3) $d_{3}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\} \leq \max _{i}\left\{\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|\right\} \leq \max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|\right\}$.
(4) $d_{4}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}\right\} \leq \max _{i}\left\{\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|^{2}\right\} \leq \max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|^{2}\right\}$.
(5) $d_{5}\left(h_{1}, h_{2}\right)=\frac{1}{2}\left(\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}\right)$

$$
\begin{aligned}
& \leq \frac{1}{2}\left(\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|+\max _{i}\left\{\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|\right\}\right) \\
& \leq \frac{1}{2}\left(\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\rho(i)}\right|\right\}\right) .
\end{aligned}
$$

(6) $d_{6}\left(h_{1}, h_{2}\right)=\frac{1}{2}\left(\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}}+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}\right\}\right)$

$$
\leq \frac{1}{2}\left(\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\varsigma(i)}-h_{2}^{\varsigma(i)}\right|^{2}}+\max _{i}\left\{\left|h_{1}^{\zeta(i)}-h_{2}^{\varsigma(i)}\right|^{2}\right\}\right)
$$

$$
\leq \frac{1}{2}\left(\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\rho(i)}-h_{2}^{\rho(i)}\right|^{2}}+\max _{i}\left\{\left|h_{1}^{\rho(i)}-h_{2}^{\rho(i)}\right|^{2}\right\}\right)
$$

Proof. Here we only prove (1) and (2), the others can be obtained similarly.
Suppose that $\pi:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ is a permutation, such that

$$
\begin{equation*}
h_{1}^{\pi(i)} \leq h_{1}^{\pi(j)}, h_{2}^{\pi(i)} \leq h_{2}^{\pi(j)}, i \leq j, i, j \neq k_{1}, k_{2} ; h_{1}^{\pi\left(k_{1}\right)} \geq h_{1}^{\pi\left(k_{2}\right)}, h_{2}^{\pi\left(k_{1}\right)} \leq h_{2}^{\pi\left(k_{2}\right)} \tag{2.62}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right| & \leq \frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\varsigma(i)}-h_{2}^{\varsigma(i)}\right|  \tag{2.63}\\
\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|^{2}} & \leq \sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\zeta(i)}-h_{2}^{\zeta(i)}\right|^{2}} \tag{2.64}
\end{align*}
$$

Based on Lemmas 2.1 and 2.3, we have

$$
\begin{align*}
& \frac{1}{l} \sum_{i=1, i \neq t_{1}, t_{2}}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|+\left|h_{1}^{\pi\left(k_{2}\right)}-h_{2}^{\pi\left(k_{1}\right)}\right|+\left|h_{1}^{\pi\left(k_{2}\right)}-h_{2}^{\pi\left(k_{1}\right)}\right| \\
& \leq \frac{1}{l} \sum_{i=1, i \neq t_{1}, t_{2}}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|+\left|h_{1}^{\pi\left(k_{1}\right)}-h_{2}^{\pi\left(k_{1}\right)}\right|+\left|h_{1}^{\pi\left(k_{2}\right)}-h_{2}^{\pi\left(k_{2}\right)}\right| \\
& \quad=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right| \tag{2.65}
\end{align*}
$$

and

$$
\begin{align*}
& \sqrt{\frac{1}{l} \sum_{i=1, i \neq \iota_{1}, t_{2}}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|^{2}+\left|h_{1}^{\pi\left(k_{2}\right)}-h_{2}^{\pi\left(k_{1}\right)}\right|^{2}+\left|h_{1}^{\pi\left(k_{1}\right)}-h_{2}^{\pi\left(k_{2}\right)}\right|^{2}} \\
& \quad \leq \sqrt{\frac{1}{l} \sum_{i=1, i \neq t_{1}, t_{2}}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|^{2}+\left|h_{1}^{\pi\left(k_{1}\right)}-h_{2}^{\pi\left(k_{1}\right)}\right|^{2}+\left|h_{1}^{\pi\left(k_{2}\right)}-h_{2}^{\pi\left(k_{2}\right)}\right|^{2}} \\
& \quad=\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\pi(i)}-h_{2}^{\pi(i)}\right|^{2}} \tag{2.66}
\end{align*}
$$

which conflicts with the assumption. With the same reason, we can prove the second part of (1).

Form Theorem 2.1, we can find that the distance measures $d_{i}\left(h_{1}, h_{2}\right)(i=1,2, \ldots, 6)$ are the smallest distance measures among the ones obtained while the orders of values in HFEs $h_{1}$ and $h_{2}$ change.

In the following, we first introduce the concept of correlation coefficient for HFEs, and then give several correlation coefficient formulas and discuss their properties:

Definition 2.4 (Xu and Xia 2011c). For two HFEs $h_{1}$ and $h_{2}$, the correlation coefficient of $h_{1}$ and $h_{2}$, denoted as $c\left(h_{1}, h_{2}\right)$, should satisfy the following properties:
(1) $\left|c\left(h_{1}, h_{2}\right)\right| \leq 1$.
(2) If $h_{1}=h_{2}$, then $c\left(h_{1}, h_{2}\right)=1$.
(3) $c\left(h_{1}, h_{2}\right)=c\left(h_{2}, h_{1}\right)$.

Based on Definition 2.4, we can construct several correlation coefficients for HFEs:

$$
\text { (1) } c_{1}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right)^{\frac{1}{2}}} \text {. }
$$

(2) $c_{2}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}}$.
(3) $c_{3}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{}$

$$
\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right)^{\frac{1}{2}}
$$

where $\bar{h}_{1}=\frac{1}{l} \sum_{i=1}^{l} h_{1}^{\sigma(i)}$ and $\bar{h}_{2}=\frac{1}{l} \sum_{i=1}^{l} h_{2}^{\sigma(i)}$.
(4) $c_{4}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}}$,
where $\bar{h}_{1}=\frac{1}{l} \sum_{i=1}^{l} h_{1}^{\sigma(i)}, \bar{h}_{2}=\frac{1}{l} \sum_{i=1}^{l} h_{2}^{\sigma(i)}$.
(5) $c_{5}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left(\frac{\Delta h^{\min }+\Delta h^{\max }}{h^{\sigma(i)}+\Delta h^{\max }}\right)$, where $\Delta h^{\sigma(i)}=\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|$,

$$
\Delta h^{\min }=\min _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}, \Delta h^{\max }=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}
$$

Although all these five formulas satisfy the properties in Definition 2.4, each of them has its own characterization. In the following, we shall discuss this issue point by point.

Theorem 2.2 (Xu and Xia 2011c). Let $h_{1}$ and $h_{2}$ be two HFEs, then

$$
\begin{aligned}
& \text { (1) } \frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\rho(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\rho(i)}\right)^{2}\right)^{\frac{1}{2}}} \leq \frac{\sum_{i=1}^{l} h_{1}^{\varsigma(i)} h_{2}^{\zeta(i)}}{\left(\sum_{i=1}^{l}\left(h_{1}^{\zeta(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\varsigma(i)}\right)^{2}\right)^{\frac{1}{2}}} \\
& \leq \frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right)^{\frac{1}{2}}}=c_{1}\left(h_{1}, h_{2}\right) \cdot \\
& \text { (2) } \frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\rho(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\rho(i)}\right)^{2}\right\}} \leq \frac{\sum_{i=1}^{l} h_{1}^{\zeta(i)} h_{2}^{\zeta(i)}}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\zeta(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\zeta(i)}\right)^{2}\right\}} \\
& \leq \frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}}=c_{2}\left(h_{1}, h_{2}\right) .
\end{aligned}
$$

Proof. Here we only prove (1), (2) can be proven similarly.
Suppose that $\pi:(1,2, \cdots, n) \rightarrow(1,2, \cdots, n)$ is a permutation satisfying

$$
\begin{equation*}
h_{1}^{\pi(i)} \leq h_{1}^{\pi(j)}, h_{2}^{\pi(i)} \leq h_{2}^{\pi(j)}, i \leq j, i, j \neq k_{1}, k_{2} ; h_{1}^{\pi\left(k_{1}\right)} \geq h_{1}^{\pi\left(k_{2}\right)}, h_{2}^{\pi\left(k_{1}\right)} \leq h_{2}^{\pi\left(k_{2}\right)} \tag{2.67}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\sum_{i=1}^{l}\left(h_{1}^{\varsigma(i)} h_{2}^{\zeta(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\zeta(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\zeta(i)}\right)^{2}\right)^{\frac{1}{2}}} \leq \frac{\sum_{i=1}^{l}\left(h_{1}^{\pi(i)} h_{2}^{\pi(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\pi(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\pi(i)}\right)^{2}\right)^{\frac{1}{2}}} \tag{2.68}
\end{equation*}
$$

By Lemma 2.2, we have

$$
\begin{equation*}
h_{1}^{\pi\left(k_{1}\right)} h_{2}^{\pi\left(k_{1}\right)}+h_{1}^{\pi\left(k_{2}\right)} h_{2}^{\pi\left(k_{2}\right)} \leq h_{1}^{\pi\left(k_{2}\right)} h_{2}^{\pi\left(k_{1}\right)}+h_{1}^{\pi\left(k_{1}\right)} h_{2}^{\pi\left(k_{2}\right)} \tag{2.69}
\end{equation*}
$$

then

$$
\begin{gather*}
\frac{\sum_{i=1}^{l}\left(h_{1}^{\pi(i)} h_{2}^{\pi(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\pi(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\pi(i)}\right)^{2}\right)^{\frac{1}{2}}}=\frac{\sum_{i=1, i \neq t_{1}, t_{2}}^{l}\left(h_{1}^{\pi(i)} h_{2}^{\pi(i)}\right)+h_{1}^{\pi\left(k_{1}\right)} h_{2}^{\pi\left(k_{1}\right)}+h_{1}^{\pi\left(k_{2}\right)} h_{2}^{\pi\left(k_{2}\right)}\left(\sum_{i=1}^{l}\left(h_{1}^{\pi(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\pi(i)}\right)^{2}\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{l}\left(h_{1}^{\pi(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\pi(i)}\right)^{2}\right)^{\frac{1}{2}}}
\end{gather*}
$$

which is inconsistent with the assumption. With the same reason, we can prove the second part of (1).

From Theorem 2.2, we can find that no matter how the orders of the values in HFEs $h_{1}$ and $h_{2}$ change, $c_{1}$ and $c_{2}$ have the highest correlation coefficients.

Theorem 2.3 (Xu and Xia, 2011b). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) $c_{2}\left(h_{1}, h_{2}\right) \leq c_{1}\left(h_{1}, h_{2}\right)$.
(2) $\left|c_{4}\left(h_{1}, h_{2}\right)\right| \leq\left|c_{3}\left(h_{1}, h_{2}\right)\right|$.

Proof. (1) Since

$$
\begin{gather*}
\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right)^{\frac{1}{2}} \leq\left(\left(\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}\right)^{2}\right)^{\frac{1}{2}} \\
=\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\} \tag{2.71}
\end{gather*}
$$

then

$$
\begin{align*}
c_{2}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}} \\
\leq c_{1}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l} h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right)^{\frac{1}{2}}} \tag{2.72}
\end{align*}
$$

(2) Since

$$
\begin{gather*}
\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right)^{\frac{1}{2}} \leq\left(\left(\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}\right)^{2}\right)^{\frac{1}{2}} \\
=\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\} \tag{2.73}
\end{gather*}
$$

and if $\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right) \geq 0$, then we have

$$
0 \leq c_{4}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}}
$$

$$
\begin{align*}
& \leq c_{3}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right)^{\frac{1}{2}}}  \tag{2.74}\\
& \text { If } \sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right) \leq 0, \text { then we have } \\
& c_{3}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right)^{\frac{1}{2}}} \\
& \leq c_{4}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\overline{h_{1}}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}} \leq 0 \tag{2.75}
\end{align*}
$$

Therefore, $\left|c_{4}\left(h_{1}, h_{2}\right)\right| \leq\left|c_{3}\left(h_{1}, h_{2}\right)\right|$.
Theorem 2.3 tells us that: (1) $c_{2}$ is always smaller than $c_{1}$, but both of them are bigger than 0 ; (2) The absolute value of $c_{4}$ is always smaller than that of $c_{3}$, and their values may be smaller or bigger than 0 , which not only provides us the strength of the relationship of HFEs, but also shows that the HFEs are positively or negatively correlated.

Theorem 2.4 (Xu and Xia 2011c). Let $h_{1}$ and $h_{2}$ be two HFEs, then
(1) If $0 \leq h_{2}^{\sigma(i)}=k h_{1}^{\sigma(i)} \leq 1$, then $c_{1}\left(h_{1}, h_{2}\right)=c_{3}\left(h_{1}, h_{2}\right)=1$ and

$$
\begin{equation*}
c_{5}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left(\frac{\min _{i}\left\{h_{1}^{\sigma(i)}\right\}+\max \left\{h_{1}^{\sigma(i)}\right\}}{h_{1}^{\sigma(i)}+\max _{i}\left\{h_{1}^{\sigma(i)}\right\}}\right) \tag{2.76}
\end{equation*}
$$

(2) Let $0 \leq h_{2}^{\sigma(i)}=k h_{1}^{\sigma(i)} \leq 1$. If $k \geq 1$, then $c_{2}\left(h_{1}, h_{2}\right)=c_{4}\left(h_{1}, h_{2}\right)=\frac{1}{k}$; If $0<k \leq 1$, then $c_{2}\left(h_{1}, h_{2}\right)=c_{4}\left(h_{1}, h_{2}\right)=k$.
(3) If $\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|=d$, then $c_{5}\left(h_{1}, h_{2}\right)=1$.

Proof. (1) If $0 \leq h_{2}^{\sigma(i)}=k h_{1}^{\sigma(i)} \leq 1, k>0$, then

$$
\begin{align*}
& c_{1}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right)^{1 / 2}}=\frac{\sum_{i=1}^{l} k\left(h_{1}^{\sigma(i)}\right)^{2}}{\sum_{i=1}^{l} k\left(h_{1}^{\sigma(i)}\right)^{2}}=1  \tag{2.77}\\
& c_{3}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\left(\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2} \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right)^{1 / 2}}=\frac{\sum_{i=1}^{l}\left(k\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}\right)}{\sum_{i=1}^{l}\left(k\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}\right)}=1 \tag{2.78}
\end{align*}
$$

and

$$
\begin{gather*}
\Delta h^{\sigma(i)}=\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|=|1-k| h_{1}^{\sigma(i)}  \tag{2.79}\\
\Delta h_{\min }=\min _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}=|1-k| \min _{i}\left\{h_{1}^{\sigma(i)}\right\}  \tag{2.80}\\
\Delta h_{\max }=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}=|1-k| \max _{i}\left\{h_{1}^{\sigma(i)}\right\} \tag{2.81}
\end{gather*}
$$

then

$$
\begin{equation*}
c_{5}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left(\frac{\Delta h^{\min }+\Delta h^{\max }}{h^{\sigma(i)}+\Delta h^{\max }}\right)=\frac{1}{l} \sum_{i=1}^{l}\left(\frac{\min _{i}\left\{h_{1}^{\sigma(i)}\right\}+\max _{i}\left\{h_{1}^{\sigma(i)}\right\}}{h_{1}^{\sigma(i)}+\max _{i}\left\{h_{1}^{\sigma(i)}\right\}}\right) \tag{2.82}
\end{equation*}
$$

(2) Let $0 \leq h_{2}^{\sigma(i)}=k h_{1}^{\sigma(i)} \leq 1$. If $k \geq 1$, then

$$
\begin{gather*}
c_{2}\left(h_{1}, h_{2}\right)=\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}}=\frac{\sum_{i=1}^{l} k\left(h_{1}^{\sigma(i)}\right)^{2}}{\sum_{i=1}^{l} k^{2}\left(h_{1}^{\sigma(i)}\right)^{2}}=\frac{1}{k}  \tag{2.83}\\
c_{4}(\alpha, \beta)=\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}}
\end{gather*}
$$

$$
\begin{equation*}
=\frac{\sum_{i=1}^{l} k\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}}{k^{2} \sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}}=\frac{1}{k} \tag{2.84}
\end{equation*}
$$

If $0<k \leq 1$, then

$$
\begin{align*}
c_{2}\left(h_{1}, h_{2}\right) & =\frac{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)} h_{2}^{\sigma(i)}\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}\right)^{2}\right\}}=\frac{\sum_{i=1}^{l} k\left(h_{1}^{\sigma(i)}\right)^{2}}{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}\right)^{2}}=k  \tag{2.85}\\
c_{4}\left(h_{1}, h_{2}\right) & =\frac{\sum_{i=1}^{l}\left(\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)\right)}{\max \left\{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}_{1}\right)^{2}, \sum_{i=1}^{l}\left(h_{2}^{\sigma(i)}-\bar{h}_{2}\right)^{2}\right\}} \\
= & \frac{\sum_{i=1}^{l}\left(k\left(h_{1}^{\sigma(i)}-\overline{h_{1}}\right)^{2}\right)}{\sum_{i=1}^{l}\left(h_{1}^{\sigma(i)}-\bar{h}\right)^{2}}=k \tag{2.86}
\end{align*}
$$

(3) If $\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|=d$, then

$$
\begin{gather*}
\Delta h^{\sigma(i)}=\left|h^{\sigma(i)}-h^{\sigma(i)}\right|=\Delta h^{\min }=\min _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\} \\
=\Delta h^{\max }=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}=d \tag{2.87}
\end{gather*}
$$

and

$$
\begin{equation*}
c_{5}\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left(\frac{\Delta h^{\min }+\Delta h^{\max }}{h^{\sigma(i)}+\Delta h^{\max }}\right)=1 \tag{2.88}
\end{equation*}
$$

From Theorem 2.4, we can conclude that:
(1) If the values of $h_{2}^{\sigma(i)}$ in $h$ are $k$ times the values of $h_{1}^{\sigma(i)}$ in $h_{1}$, then the correlation coefficients $c_{1}$ and $c_{3}$ are $1, c_{2}$ and $c_{4}$ are $\frac{1}{k}(k \geq 1)$ or $k(0<k \leq 1)$.
(2) If $\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|=d$, then $c_{5}\left(h_{1}, h_{2}\right)=1$.

This indicates that these five correlation coefficient formulas reflect different linear relationships between two HFEs $h_{1}$ and $h_{2}$, and therefore, they may produce different results for the same two HFEs, which is reasonable.

In what follows, we use an example to illustrate the developed correlation coefficient formulas:

Example 2.3. (Szmidt and Kacprzyk 2004) To make a proper diagnosis $A=\{$ Viral fever, Malaria, Typhoid, Stomach problem, Chest problem \} for a patient with the given values of the symptoms, $X=\{$ Temperature, headache, cough, stomach pain, chest pain\}, a medical knowledge base is necessary that involves elements described in terms of HFSs. The data are given in Table 2.6 ( Xu and Xia 2011c), and each symptom is described by a HFE). The set of patients is $P=\{\mathrm{Al}, \mathrm{Bob}, \mathrm{Joe}, \mathrm{Ted}\}$. The symptoms are given in Table 2.7 ( Xu and Xia 2011c). We need to seek a diagnosis for each patient.

Table 2.6. Symptoms characteristic for the considered diagnoses

|  | Temperature | Headache | Cough | Stomach pain | Chest pain |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Viral fever | $(0.6,0.4,0.3)$ | $(0.7,0.5,0.3,0.2)$ | $(0.5,0.3)$ | $(0.5,0.4,0.3,0.2,0.1)$ | $(0.5,0.4,0.2,0.1)$ |
| Malaria | $(0.9,0.8,0.7)$ | $(0.5,0.3,0.2,0.1)$ | $(0.2,0.1)$ | $(0.6,0.5,0.3,0.2,0.1)$ | $(0.4,0.3,0.2,0.1)$ |
| Typhoid | $(0.6,0.3,0.1)$ | $(0.9,0.8,0.7,0.6)$ | $(0.5,0.3)$ | $(0.5,0.4,0.3,0.2,0.1)$ | $(0.6,0.4,0.3,0.1)$ |
| Stomach problem | $(0.5,0.4,0.2)$ | $(0.4,0.3,0.2,0.1)$ | $(0.4,0.3)$ | $(0.9,0.8,0.7,0.6,0.5)$ | $(0.5,0.4,0.2,0.1)$ |
| Chest Problem | $(0.3,0.2,0.1)$ | $(0.5,0.3,0.2,0.1)$ | $(0.3,0.2)$ | $(0.7,0.6,0.5,0.3,0.2)$ | $(0.9,0.8,0.7,0.6)$ |

Table 2.7. Symptoms characteristic for the considered patients

|  | Temperature | Headache | Cough | Stomach pain | Chest pain |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | $(0.9,0.7,0.5)$ | $(0.4,0.3,0.2,0.1)$ | $(0.4,0.3)$ | $(0.6,0.5,0.4,0.2,0.1)$ | $(0.4,0.3,0.2,0.1)$ |
| Bob | $(0.5,0.4,0.2)$ | $(0.5,0.4,0.3,0.1)$ | $(0.2,0.1)$ | $(0.9,0.8,0.6,0.5,0.4)$ | $(0.5,0.4,0.3,0.2)$ |
| Joe | $(0.9,0.7,0.6)$ | $(0.7,0.4,0.3,0.1)$ | $(0.3,0.2)$ | $(0.6,0.4,0.3,0.2,0.1)$ | $(0.6,0.3,0.2,0.1)$ |
| Ted | $(0.8,0.7,0.5)$ | $(0.6,0.5,0.4,0.2)$ | $(0.4,0.3)$ | $(0.6,0.4,0.3,0.2,0.1)$ | $(0.5,0.4,0.2,0.1)$ |

We utilize the correlation coefficient $c_{1}$ to derive a diagnosis for each patient. All the results for the considered patients are listed in Table 2.8 ( Xu and Xia 2011c). From the arguments in Table 2.8, we can find that A1 and Ted suffer from Viral fever, Bob from stomach problem, and Joe from Malaria.

Table 2.8. The values of $c_{1}$ for each patient to the considered set of possible diagnoses

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest Problem |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | $\mathbf{0 . 9 9 6 9}$ | 0.9929 | 0.9800 | 0.9902 | 0.9878 |
| Bob | 0.9900 | 0.9862 | 0.9792 | $\mathbf{0 . 9 9 2 1}$ | 0.9909 |
| Joe | 0.9927 | $\mathbf{0 . 9 9 2 9}$ | 0.9677 | 0.9817 | 0.9750 |
| Ted | $\mathbf{0 . 9 9 4 2}$ | 0.9899 | 0.9787 | 0.9879 | 0.9772 |

If we utilize the correlation coefficient formulas $c_{2}, c_{3}, c_{4}$ and $c_{5}$ to derive a diagnosis, then the results are listed in Tables 2.9-2.12, respectively.

Table 2.9. The values of $c_{2}$ for each patient to the considered set of possible diagnoses

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest Problem |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | 0.7373 | $\mathbf{0 . 8 2 9 9}$ | 0.6564 | 0.7770 | 0.6099 |
| Bob | 0.6784 | 0.7280 | 0.6403 | $\mathbf{0 . 8 0 8 3}$ | 0.6501 |
| Joe | $\mathbf{0 . 7 9 7 7}$ | 0.7903 | 0.6864 | 0.6570 | 0.6306 |
| Ted | $\mathbf{0 . 8 7 5 1}$ | 0.7394 | 0.7471 | 0.7313 | 0.5712 |

Table 2.10. The values of $c_{3}$ for each patient to the considered set of possible diagnoses

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest Problem |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | 0.9906 | 0.9919 | 0.9954 | 0.9926 | $\mathbf{0 . 9 9 6 5}$ |
| Bob | 0.9712 | 0.9850 | 0.9840 | $\mathbf{0 . 9 9 2 8}$ | 0.9803 |
| Joe | 0.9786 | 0.9823 | $\mathbf{0 . 9 8 5 7}$ | 0.9652 | 0.9798 |
| Ted | 0.9723 | 0.9785 | 0.9775 | $\mathbf{0 . 9 9 3 8}$ | 0.9760 |

Table 2.11. The values of $c_{4}$ for each patient to the considered set of possible diagnoses

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest Problem |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | 0.6564 | 0.8439 | 0.7321 | 0.8412 | $\mathbf{0 . 8 4 8 6}$ |
| Bob | 0.7226 | $\mathbf{0 . 9 1 7 2}$ | 0.6386 | 0.8397 | 0.9125 |
| Joe | 0.7926 | 0.7603 | 0.6703 | $\mathbf{0 . 8 0 6 4}$ | 0.7579 |
| Ted | 0.7936 | 0.8386 | 0.6958 | $\mathbf{0 . 9 1 0 7}$ | 0.8362 |

Table 2.12. The values of $c_{5}$ for each patient to the considered set of possible diagnoses

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest Problem |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | 0.7911 | 0.8994 | 0.7150 | 0.9317 | $\mathbf{0 . 9 5 7 8}$ |
| Bob | 0.7983 | $\mathbf{0 . 9 4 3 3}$ | 0.9183 | 0.8150 | 0.8767 |
| Joe | 0.8550 | 0.8300 | 0.8470 | 0.8633 | $\mathbf{0 . 8 8 6 1}$ |
| Ted | 0.8689 | 0.8883 | 0.8404 | $\mathbf{0 . 9 4 2 5}$ | 0.9317 |

From Tables 2.9-2.12, we know that the results obtained by different correlation coefficient formulas are different. That is because these correlation coefficient formulas are based on different linear relationships, and may produce different results, which has been mentioned in the existing literature (Merigó and Casanovas 2009; Chen and Li 2010).

### 2.3 Hesitant Fuzzy Entropy and Cross-Entropy and Their Use in MADM

In the following, we first introduce the axiomatic definition of entropy for HFEs:
Definition 2.5 (Xu and Xia 2012a). An entropy on a HFE $h$ is a real-valued function $E: H \rightarrow[0,1]$, satisfying the following axiomatic requirements:
(1) $E(h)=0$, if and only if $h=\{0\}$ or $h=\{1\}$.
(2) $E(h)=1$, if and only if $h_{\sigma(i)}+h_{\sigma\left(h_{h}-i+1\right)}=1$ for $i=1,2, \cdots, l_{h}$.
(3) $E\left(h_{1}\right) \leq E\left(h_{2}\right)$, if $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \leq 1$ or $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \geq 1, i=1,2, \cdots, l$.
(4) $E(h)=E\left(h^{c}\right)$.

Motivated by the entropy measures for fuzzy sets (Fan 2002; Parkash et al. 2008), we can construct some entropy formulas based on Definition 2.5 as follows:

$$
\begin{align*}
& E_{1}(h)= \frac{1}{l_{h}(\sqrt{2}-1)} \sum_{i=1}^{l_{h}}\left(\sin \frac{\pi\left(h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}\right)}{4}+\sin \frac{\pi\left(2-h^{\sigma(i)}-h^{\sigma\left(l_{h}-i+1\right)}\right)}{4}-1\right)  \tag{2.89}\\
& E_{2}(h)= \frac{1}{l_{h}(\sqrt{2}-1)} \sum_{i=1}^{l_{h}}\left(\cos \frac{\pi\left(h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}\right)}{4}+\cos \frac{\pi\left(2-h^{\sigma(i)}-h^{\sigma\left(l_{h}-i+1\right)}\right)}{4}-1\right)  \tag{2.90}\\
& E_{3}(h)=-\frac{1}{l_{h} \ln 2} \sum_{i=1}^{l_{h}}\left(\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2} \ln \frac{h^{\sigma(i)}+h^{\sigma(l-i+1)}}{2}\right. \\
&\left.+\frac{2-h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2} \ln \frac{2-h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)  \tag{2.91}\\
& E_{4}(h)= \frac{1}{l_{h}\left(2^{\left(1-\lambda_{1}\right) \lambda_{2}}-1\right)} \sum_{i=1}^{l_{h}}\left(\left(\left(\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}+\left(1-\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}\right)^{\lambda_{2}}-1\right), \\
& \lambda_{2} \neq 0, \lambda_{1} \neq 1, \lambda_{1}>0 \tag{2.92}
\end{align*}
$$

Moreover, with the change of the parameters in $E_{4}$, some special cases can be obtained (Xu and Xia 2012a):

If $\lambda_{2}=1$, then
$E_{4}(h)=\frac{1}{l_{h}\left(2^{1-\lambda_{1}}-1\right)} \sum_{i=1}^{l_{h}}\left(\left(\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}+\left(1-\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}-1\right)$
If $\lambda_{2}=\frac{1}{\lambda_{1}}$, then

$$
\begin{equation*}
E_{4}(h)=\frac{\lambda_{1}}{l_{h}\left(2^{\frac{1-\lambda_{1}}{\lambda_{1}}}-1\right)} \sum_{i=1}^{l_{h}}\left(\left(\left(\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}+\left(1-\frac{h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda_{1}}\right)^{\frac{1}{\lambda_{1}}}-1\right) \tag{2.94}
\end{equation*}
$$

Xu and Xia (2011b) gave the hesitant fuzzy similarity measure defined as:
Definition 2.6 (Xu and Xia 2011c). For two HFEs $h_{1}$ and $h_{2}$, the similarity measure between $h_{1}$ and $h_{2}$, denoted as $\bar{s}\left(h_{1}, h_{2}\right)$, should satisfy the following properties:
(1) $\bar{s}\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}=\{0\}, h_{2}=\{1\}$ or $h_{1}=\{1\}, h_{2}=\{0\}$.
(2) $\bar{s}\left(h_{1}, h_{2}\right)=1$ if and only if $h_{1}^{\sigma(i)}=h_{2}^{\sigma(i)}, i=1,2, \cdots, l$.
(3) $\bar{s}\left(h_{1}, h_{3}\right) \leq \bar{s}\left(\left(h_{1}, h_{2}\right), \bar{s}\left(h_{1}, h_{3}\right) \leq \bar{s}\left(h_{2}, h_{3}\right)\right.$, if $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq h_{3}^{\sigma(i)}$ or $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)} \geq h_{3}^{\sigma(i)}, i=1,2, \cdots, l$.
(4) $\bar{s}\left(h_{1}, h_{2}\right)=\bar{s}\left(h_{2}, h_{1}\right)$.

Based on Definition 2.6, some hesitant fuzzy similarity measures can be constructed as (Xu and Xia 2011c):

$$
\begin{gather*}
\bar{s}_{1}\left(h_{1}, h_{2}\right)=1-\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|  \tag{2.95}\\
\bar{s}_{2}\left(h_{1}, h_{2}\right)=1-\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}}  \tag{2.96}\\
\bar{s}_{3}\left(h_{1}, h_{2}\right)=1-\lambda \sqrt[\lambda]{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{\lambda}}  \tag{2.97}\\
\bar{s}_{4}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}  \tag{2.98}\\
\bar{s}_{5}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}\right\}  \tag{2.99}\\
\bar{s}_{6}\left(h_{1}, h_{2}\right)=\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{\lambda}\right\} \tag{2.100}
\end{gather*}
$$

$$
\begin{align*}
& \bar{s}_{7}\left(h_{1}, h_{2}\right)=1-\frac{1}{2}\left(\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|\right\}\right)  \tag{2.101}\\
& \bar{s}_{8}\left(h_{1}, h_{2}\right)=1-\frac{1}{2}\left(\sqrt{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}}+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{2}\right\}\right)  \tag{2.102}\\
& \bar{s}_{9}\left(h_{1}, h_{2}\right)=1-\frac{1}{2}\left(\sqrt[\lambda]{\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{\lambda}}+\max _{i}\left\{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|^{\lambda}\right\}\right) \tag{2.103}
\end{align*}
$$

where $\lambda>0$.
By analyzing these similarity measures, we can find that $\bar{s}_{1}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{2}\left(h_{1}, h_{2}\right)$ are based on Hamming distance and Euclidean distance; $\bar{s}_{4}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{5}\left(h_{1}, h_{2}\right)$ apply Hausdorff metric to $\bar{s}_{1}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{2}\left(h_{1}, h_{2}\right)$; $\bar{s}_{7}\left(h_{1}, h_{2}\right)$ combines $\bar{s}_{1}\left(h_{1}, h_{2}\right)$ with $\bar{s}_{4}\left(h_{1}, h_{2}\right) ; \quad \bar{s}_{8}\left(h_{1}, h_{2}\right)$ combines $\bar{s}_{2}\left(h_{1}, h_{2}\right)$ with $\bar{s}_{5}\left(h_{1}, h_{2}\right) ; \bar{s}_{3}\left(h_{1}, h_{2}\right), \bar{s}_{6}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{9}\left(h_{1}, h_{2}\right)$ are further generalizations of $\bar{s}_{1}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{2}\left(h_{1}, h_{2}\right), \quad \bar{s}_{4}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{5}\left(h_{1}, h_{2}\right), \bar{s}_{7}\left(h_{1}, h_{2}\right)$ and $\bar{s}_{8}\left(h_{1}, h_{2}\right)$, respectively; When $\lambda=1$, then $\bar{s}_{3}\left(h_{1}, h_{2}\right)$ becomes $\bar{s}_{1}\left(h_{1}, h_{2}\right), \quad \bar{s}_{6}\left(h_{1}, h_{2}\right)$ becomes $\bar{s}_{4}\left(h_{1}, h_{2}\right)$, and $\bar{s}_{9}\left(h_{1}, h_{2}\right)$ becomes $\bar{s}_{7}\left(h_{1}, h_{2}\right)$; When $\lambda=2$, then $\bar{s}_{3}\left(h_{1}, h_{2}\right)$ reduces to $\bar{s}_{2}\left(h_{1}, h_{2}\right), \bar{s}_{6}\left(h_{1}, h_{2}\right)$ reduces to $\bar{s}_{5}\left(h_{1}, h_{2}\right)$, and $\bar{s}_{9}\left(h_{1}, h_{2}\right)$ reduces to $\bar{s}_{8}\left(h_{1}, h_{2}\right)$.
Many authors have investigated the relationships between similarity measures and entropy formulas under different environments, such as interval-valued fuzzy sets (Zeng and Guo 2008; Zeng and Li 2006), interval-valued intuitionistic fuzzy sets (Zhang et al. 2009; Zhang et al. 2010). In what follows, we discuss the relationships between hesitant fuzzy similarity measures and hesitant fuzzy entropy formulas:

Theorem 2.5 (Xu and Xia 2012a). Let $h$ be a HFE, then $\bar{s}\left(h, h^{c}\right)$ ) is the entropy of $h$.

Proof. (1) $\bar{s}\left(h, h^{c}\right)=0 \Leftrightarrow h=\{0\}$ and $h^{c}=\{1\}$ or $h^{c}=\{0\}$ and $h=\{1\}$.
(2) $\bar{s}\left(h, h^{c}\right)=1 \Leftrightarrow h=h^{c} \Leftrightarrow h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}=1$, for $i=1,2, \cdots, l_{h}$.
(3) Suppose that $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)}$, for $h_{2}^{\sigma(i)}+h_{2}^{\sigma\left(l_{h}-i+1\right)} \leq 1, i=1,2, \cdots, l$, then $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq 1-h_{2}^{\sigma\left(l_{h}-i+1\right)} \leq 1-h_{1}^{\sigma\left(l_{h}-i+1\right)}$. Therefore, known by the definition of the similarity measure of HFE, we have $\bar{s}\left(h_{1}, h_{1}^{c}\right) \leq \bar{s}\left(h_{2}, h_{1}^{c}\right) \leq \bar{s}\left(h_{2}, h_{2}^{c}\right)$. With the same reason, we can prove it when $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma\left(l_{h}-i+1\right)} \geq 1, i=1,2, \cdots, l$.
(4) $\bar{s}\left(h, h^{c}\right)=\bar{s}\left(h^{c}, h\right)$.

Example 2.4 (Xu and Xia 2012a). For two HFEs $h_{1}$ and $h_{2}, \lambda>0$, we can construct the following entropy formulas based on the similarity measures (2.95)-(2.103):

$$
\begin{gather*}
\bar{s}_{1}\left(h, h^{c}\right)=1-\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|  \tag{2.104}\\
\bar{s}_{2}\left(h, h^{c}\right)=1-\sqrt{\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}}  \tag{2.105}\\
\bar{s}_{3}\left(h, h^{c}\right)=1-\lambda \sqrt{\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}}  \tag{2.106}\\
\bar{s}_{4}\left(h, h^{c}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|\right\}  \tag{2.107}\\
\bar{s}_{5}\left(h, h^{c}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}\right\}  \tag{2.108}\\
\bar{s}_{6}\left(h, h^{c}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}\right\} \tag{2.109}
\end{gather*}
$$

$\bar{s}_{7}\left(h, h^{c}\right)=1-\frac{1}{2}\left(\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|\right\}\right)$
$\bar{S}_{8}\left(h, h^{c}\right)=1-\frac{1}{2}\left(\sqrt{\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}}+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}\right\}\right)$

$$
\begin{equation*}
\bar{s}_{9}\left(h, h^{c}\right)=1-\frac{1}{2}\left(\sqrt[\lambda]{\frac{1}{l_{h}} \sum_{i=1}^{l_{h}}\left|h^{\sigma(i)}+h^{\sigma\left(h_{h}-i+1\right)}-1\right|^{2}}+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(h_{h}-i+1\right)}-1\right|^{2}\right\}\right) \tag{2.112}
\end{equation*}
$$

In this paper, we let $\left[\frac{l_{h}}{2}\right]$ denote the largest integer no bigger than $\frac{l_{h}}{2}$, and $\left[\overline{l_{h}} \overline{2}\right]$ denote the smallest integer no smaller than $\frac{l_{h}}{2}$, then we get the following theorem:

Theorem 2.6 (Xu and Xia 2012a). For a HFE $h$, let $H_{1}=\left\{h^{\sigma(1)}, h^{\sigma(2)}, \cdots, h^{\sigma\left[\frac{h}{2}\right]}\right\}$ and $\left.\quad H_{2}=\left\{1-h^{\sigma\left(l_{h}\right)}, 1-h^{\sigma\left(l_{h}-1\right)}, \cdots, 1-h^{\sigma\left(\overline{\left[\frac{l_{h}}{2}\right]}+1\right.}\right)\right\}$, then $\bar{s}\left(H_{1}, H_{2}\right)$ is the entropy of $h$.

Proof. (1) $\bar{s}\left(H_{1}, H_{2}\right)=0 \Leftrightarrow H_{1}=\{0\}, H_{2}=\{1\} \quad$ or $H_{1}=\{1\}$, $H_{2}=\{0\} \Leftrightarrow h=\{0\}$ or $h=\{1\}$.
(2) $\bar{s}\left(H_{1}, H_{2}\right)=1 \Leftrightarrow H_{1}=H_{2}$, for $i=1,2, \cdots, l_{h}$.
(3) Assume $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma\left(l_{h}-i+1\right)} \leq 1, i=1,2, \cdots, l_{h}$, then we have

$$
\begin{equation*}
h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq 1-h_{2}^{\sigma\left(l_{h}-i+1\right)} \leq 1-h_{1}^{\sigma\left(l_{h}-i+1\right)} \tag{2.113}
\end{equation*}
$$

Therefore, known by the definition of the similarity measure of HFEs, we have $\bar{s}\left(h_{1}^{\sigma(i)}, 1-h_{1}^{\sigma\left(l_{h}-i+1\right)}\right) \leq \bar{s}\left(h_{2}^{\sigma(i)}, 1-h_{1}^{\sigma\left(l_{h}-i+1\right)}\right) \leq \bar{s}\left(h_{2}^{\sigma(i)}, 1-h_{2}^{\sigma\left(l_{h}-i+1\right)}\right)$

Similarly, we can prove it is also true that $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)}$, for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \geq 1, i=1,2, \cdots, l_{h}$.
(4) $\bar{s}\left(H_{1}, H_{2}\right)=\bar{s}\left(H_{2}, H_{1}\right)$.

Example 2.5 (Xu and Xia 2012a). For a HFE $h$, we can construct the following entropy formulas based on the similarity measures (2.95)-(2.103):

$$
\begin{align*}
& \bar{s}_{1}\left(H_{1}, H_{2}\right)=1-\left[\frac{2}{l_{h}}\right] \overline{\sum_{i=1}^{\left[\frac{l_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right| \\
& \bar{s}_{2}\left(H_{1}, H_{2}\right)=1-\sqrt{\left[\frac{2}{l_{h}}\right] \sum_{i=1}^{\overline{\left[\frac{l_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}} \\
& \bar{S}_{3}\left(H_{1}, H_{2}\right)=1-\sqrt[\lambda]{\left[\frac{2}{\left.l_{h}\right]} \overline{\sum_{i=1}^{\left[\frac{l_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}\right.} \\
& \bar{s}_{4}\left(H_{1}, H_{2}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|\right\} \\
& \bar{s}_{5}\left(H_{1}, H_{2}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}\right\} \\
& \bar{s}_{6}\left(H_{1}, H_{2}\right)=\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}\right\} \\
& \bar{s}_{7}\left(H_{1}, H_{2}\right)=1-\left[\frac{1}{l_{h}}\right]\left(\overline{\sum_{i=1}^{\left[\frac{h_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|\right\}\right)  \tag{2.121}\\
& \bar{s}_{8}\left(H_{1}, H_{2}\right)=1-\left[\frac{1}{l_{h}}\right]\left(\sqrt{\sum_{i=1}^{\left[\frac{l_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{2}\right\}\right)  \tag{2.122}\\
& \left.\bar{s}_{9}\left(H_{1}, H_{2}\right)=1-\overline{\left[\frac{1}{l_{h}}\right.}\right]\left(\sqrt[\lambda]{\left.\overline{\sum_{i=1}^{\left[\frac{l_{h}}{2}\right.}}\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}+\max _{i}\left\{\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|^{\lambda}\right\}\right)}\right. \tag{2.123}
\end{align*}
$$

For two HFEs $h_{1}$ and $h_{2}$, suppose that $\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}<\left|h_{1}^{\sigma(i+1)}-h_{2}^{\sigma(i+1)}\right|\right.$, $i=1,2, \cdots, l-1$, and

$$
\begin{equation*}
f\left(h_{1}, h_{2}\right)=\left(\frac{\left|h_{1}^{\sigma(1)}-h_{2}^{\sigma(1)}\right|+1}{2}, \frac{\left|h_{1}^{\sigma(2)}-h_{2}^{\sigma(2)}\right|+1}{2}, \cdots, \frac{\left|h_{1}^{\sigma(l)}-h_{2}^{\sigma(1)}\right|+1}{2}\right)( \tag{2.124}
\end{equation*}
$$

then we have the following theorem:
Theorem 2.7 (Xu and Xia 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, and $E$ the entropy of a HFE, then $E\left(f\left(h_{1}, h_{2}\right)\right)$ is the similarity measure of the HFEs $h_{1}$ and $h_{2}$.

Proof. (1) $E\left(f\left(h_{1}, h_{2}\right)\right)=0 \Leftrightarrow f\left(h_{1}, h_{2}\right)=1$ or $f\left(h_{1}, h_{2}\right)=0 \Leftrightarrow h_{1}=\{0\}$, $h_{2}=\{1\}$ or $h_{1}=\{1\}, h_{2}=\{0\}$.
(2) $E\left(f\left(h_{1}, h_{2}\right)\right)=1 \Leftrightarrow \frac{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+1}{2}+\frac{\left|h_{1}^{\sigma(i+1)}-h_{2}^{\sigma(i+1)}\right|+1}{2}=1 \Leftrightarrow h_{1}=h_{2}$.
(3) Suppose that three HFEs $h_{1}, h_{2}$ and $h_{3}$ have the same length $l$. Since $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq h_{3}^{\sigma(i)}, i=1,2, \cdots, l$, then we can obtain

$$
\begin{equation*}
\frac{\left|h_{1}^{\sigma(i)}-h_{3}^{\sigma(i)}\right|+1}{2} \geq \frac{\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+1}{2}, i=1,2, \cdots, l \tag{2.125}
\end{equation*}
$$

which implies $f\left(h_{1}, h_{3}\right) \geq f\left(h_{1}, h_{2}\right)$. From the definition of $f\left(h_{1}, h_{2}\right)$, we know that

$$
\begin{equation*}
\left(f\left(h_{1}, h_{2}\right)\right)^{\sigma(i)}+\left(f\left(h_{1}, h_{2}\right)\right)^{\sigma(l-i+1)} \geq 1, i=1,2, \cdots, l \tag{2.126}
\end{equation*}
$$

thus $E\left(f\left(h_{1}, h_{3}\right)\right) \leq E\left(f\left(h_{1}, h_{2}\right)\right)$. With the same reason, we can prove it is also true for $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)} \geq h_{3}^{\sigma(i)}, \quad i=1,2, \cdots, l$.
(4) $E\left(f\left(h_{1}, h_{2}\right)\right)=E\left(f\left(h_{2}, h_{1}\right)\right)$.

Example 2.6 (Xu and Xia 2012a). For two HFEs $h_{1}$ and $h_{2}$, we have

$$
E_{1}\left(f\left(h_{1}, h_{2}\right)\right)=\frac{1}{l(\sqrt{2}-1)} \sum_{i=1}^{l}\left(\sin \frac{\pi\left(2+\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|\right)}{8}\right.
$$

$$
\begin{gather*}
\left.+\sin \frac{\pi\left(2-\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|-\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|\right)}{8}-1\right)  \tag{2.127}\\
E_{2}\left(f\left(h_{1}, h_{2}\right)\right)=\frac{1}{l(\sqrt{2}-1)} \sum_{i=1}^{l}\left(\cos \frac{\pi\left(2+\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|\right)}{8}\right. \\
\left.+\cos \frac{\pi\left(2-\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|-\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|\right)}{8}-1\right) \\
E_{3}\left(f\left(h_{1}, h_{2}\right)\right)=-\frac{1}{l \ln 2} \sum_{i=1}^{l}\left(\frac{2+\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|}{4}\right. \\
\times \ln \frac{2+\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|}{4} \\
\times \frac{2-\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|-\mid h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1) \mid}}{4} \\
\left.\times \ln \frac{2-\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|-\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|}{4}\right) \tag{2.129}
\end{gather*}
$$

$E_{4}\left(f\left(h_{1}, h_{2}\right)\right)$

$$
=-\frac{1}{l\left(2^{\left(1-\lambda_{1}\right) \lambda_{2}}-1\right)} \sum_{i=1}^{l}\left(\left(\left(\frac{2+\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|+\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|}{4}\right)^{\lambda_{1}}\right.\right.
$$

$$
\begin{equation*}
\left.\left.+\left(\frac{2-\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|-\left|h_{1}^{\sigma(l-i+1)}-h_{2}^{\sigma(l-i+1)}\right|}{4}\right)^{\lambda_{1}}\right)^{\lambda_{2}}-1\right), \lambda_{2} \neq 0, \lambda_{1} \neq 1, \lambda_{1}>0 \tag{2.130}
\end{equation*}
$$

Corollary 2.1 (Xu and Xia 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, and $E$ the entropy of HFE, then $E\left(\left(f\left(h_{1}, h_{2}\right)\right)^{c}\right)$ is the similarity measure of HFEs $h_{1}$ and $h_{2}$.

For two HFEs $h_{1}$ and $h_{2}$, suppose that

$$
\begin{equation*}
\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right|<\left|h_{1}^{\sigma(i+1)}-h_{2}^{\sigma(i+1)}\right|, \quad i=1,2, \cdots, l-1 \tag{2.131}
\end{equation*}
$$

then we define

$$
\begin{equation*}
g\left(h_{1}, h_{2}\right)=\left\{\frac{\left|h_{1}^{\sigma(1)}-h_{2}^{\sigma(1)}\right|^{\lambda}+1}{2}, \frac{\left|h_{1}^{\sigma(2)}-h_{2}^{\sigma(2)}\right|^{\lambda}+1}{2}, \cdots, \frac{\left|h_{1}^{\sigma(l)}-h_{2}^{\sigma(l)}\right|^{\lambda}+1}{2}\right\}, \lambda>0 \tag{2.132}
\end{equation*}
$$

from which we get
Corollary 2.2 (Xu and Xia 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, and $E$ the entropy of HFE, then $E\left(g\left(h_{1}, h_{2}\right)\right)$ is the similarity measure of HFEs $h_{1}$ and $h_{2}$.

For a HFE $h$, suppose that

$$
\begin{equation*}
\left|h^{\sigma(i)}+h^{\sigma(l-i+1)}-1\right|<\left|h^{\sigma(i+1)}+h^{\sigma(l-i)}-1\right|, \quad i=1,2, \cdots,\left[\frac{l_{h}}{2}\right] \tag{2.133}
\end{equation*}
$$

we define two HFEs $\hat{m}(h)$ and $\hat{n}(h)$ :

$$
\begin{equation*}
\hat{m}(h)=\left\{\frac{1+\left|h^{\sigma(1)}+h^{\sigma\left(l_{h}\right)}-1\right|}{2}, \frac{1+\left|h^{\sigma(2)}+h^{\sigma\left(l_{h}-1\right)}-1\right|}{2}, \cdots, \left.\frac{\left.1+\left\lvert\, h^{\sigma\left(\overline{\left[\frac{h_{h}}{2}\right]}\right)}+h^{\sigma\left(h_{h}-\overline{\left[\frac{l_{h}}{2}\right]}+1\right.}\right.\right)}{2}-1 \right\rvert\,\right\} \tag{2.134}
\end{equation*}
$$

$\hat{n}(h)=\left\{\frac{1-\left|h^{\sigma(1)}+h^{\sigma\left(l_{h}\right)}-1\right|}{2}, \frac{1-\left|h^{\sigma(2)}+h^{\sigma\left(l_{h}-1\right)}-1\right|}{2}, \cdots, \frac{\left.1-\left\lvert\, h^{\sigma\left[\overline{\left[\frac{l_{h}}{2}\right]}\right)}+h^{\sigma\left(\overline{\left.l_{h}-\overline{l_{h}}\right]}+1\right.}\right.\right)}{2}\right\}$
then we have the following theorem:
Theorem 2.8 (Xu and Xia 2012a). Suppose that $\bar{s}$ is the similarity measure for HFEs, then $\bar{s}(\hat{m}(h), \hat{n}(h))$ is the entropy of the HFE $h$.

Proof. (1) $\bar{s}(\hat{m}(h), \hat{n}(h))=0 \Leftrightarrow \hat{m}(h)=\{1\} \quad$ and $\quad \hat{n}(h)=\{0\} \quad$ or $\hat{m}(h)=\{0\}$ and $\hat{n}(h)=\{1\} \Leftrightarrow h=\{1\}$ or $h=\{0\}$.
(2) $\bar{s}(\hat{m}(h), \hat{n}(h))=1 \Leftrightarrow \frac{1+\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|}{2}=\frac{1-\left|h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}-1\right|}{2}$

$$
\begin{aligned}
& \Leftrightarrow h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}=1, i=1,2, \cdots,\left[\frac{l_{h}}{2}\right] \\
& \Leftrightarrow h^{\sigma(i)}+h^{\sigma\left(l_{h}-i+1\right)}=1, \text { for } i=1,2, \cdots, l_{h} .
\end{aligned}
$$

(3) Since $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \leq 1, i=1,2, \cdots, l$ which implies

$$
\begin{equation*}
h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq 1-h_{2}^{\sigma(l-i+1)} \leq 1-h_{1}^{\sigma(l-i+1)} \tag{2.136}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|h_{1}^{\sigma(i)}+h_{1}^{\sigma(l-i+1)}-1\right| \geq\left|h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)}-1\right|, \quad i=1,2, \cdots, l \tag{2.137}
\end{equation*}
$$

which means that

$$
\begin{equation*}
\hat{n}\left(h_{1}^{\sigma(i)}\right) \subseteq \hat{n}\left(h_{2}^{\sigma(i)}\right) \subseteq \hat{m}\left(\dot{h}_{2}^{\sigma(i)}\right) \subseteq \hat{m}\left(h_{1}^{\sigma(i)}\right), \quad i=1,2, \cdots, l \tag{2.138}
\end{equation*}
$$

Therefore, from the definition of the similarity measure of HFE, we have

$$
\begin{equation*}
\bar{s}\left(\hat{m}\left(h_{1}\right), \hat{n}\left(h_{1}\right)\right) \leq \bar{s}\left(\hat{m}\left(h_{2}\right), \hat{n}\left(h_{1}\right)\right) \leq \bar{s}\left(\hat{m}\left(h_{2}\right), \hat{n}\left(h_{2}\right)\right) \tag{2.139}
\end{equation*}
$$

With the same reason, when $h_{1}^{\sigma(i)} \geq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \geq 1$, $i=1,2, \cdots, l$, we can also prove $\bar{s}\left(\hat{m}\left(h_{1}\right), \hat{n}\left(h_{1}\right)\right) \leq \bar{s}\left(\hat{m}\left(h_{2}\right), \hat{n}\left(h_{2}\right)\right)$.
(4) $\bar{s}(\hat{m}(h), \hat{n}(h))=\bar{s}\left(\hat{m}\left(h^{c}\right), \hat{n}\left(h^{c}\right)\right)$.

Corollary 2.3 (Xu and Xia 2012a). Suppose that $\bar{s}$ is a similarity measure for HFEs, then $\bar{s}\left((\hat{m}(h))^{c},(\hat{n}(h))^{c}\right)$ is the entropy of the HFE $h$.

Example 2.7 (Xu and Xia 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, we have

$$
\begin{aligned}
& \bar{s}_{1}(\hat{m}(h), \hat{n}(h))=\bar{s}_{1}\left(H_{1}, H_{2}\right), \bar{s}_{2}(\hat{m}(h), \hat{n}(h))=\bar{s}_{2}\left(H_{1}, H_{2}\right) \\
& \bar{s}_{3}(\hat{m}(h), \hat{n}(h))=\bar{s}_{3}\left(H_{1}, H_{2}\right), \bar{s}_{4}(\hat{m}(h), \hat{n}(h))=\bar{s}_{4}\left(H_{1}, H_{2}\right)
\end{aligned}
$$

$$
\begin{gather*}
\bar{s}_{5}(\hat{m}(h), \hat{n}(h))=\bar{s}_{5}\left(H_{1}, H_{2}\right), \bar{s}_{6}(\hat{m}(h), \hat{n}(h))=\bar{s}_{6}\left(H_{1}, H_{2}\right) \\
\bar{s}_{7}(\hat{m}(h), \hat{n}(h))=\bar{s}_{7}\left(H_{1}, H_{2}\right), \bar{s}_{8}(\hat{m}(h), \hat{n}(h))=\bar{s}_{8}\left(H_{1}, H_{2}\right) \\
\bar{s}_{9}(\hat{m}(h), \hat{n}(h))=\bar{s}_{9}\left(H_{1}, H_{2}\right) \tag{2.140}
\end{gather*}
$$

Xu and Xia (2012a) introduced the axiomatic definition of cross-entropy measure for HFEs motivated by Shang and Jiang (1997), Vlachos and Sergiadis (2007), Hung and Yang (2008), from which we can also get some entropy measures for HFEs.

According to Shannon's inequality (Lin 1991), we first give the following definition:

Definition 2.7 (Xu and Xia 2012a). Let $h_{1}$ and $h_{2}$ be two HFEs, then the cross-entropy $C\left(h_{1}, h_{2}\right)$ of $h_{1}$ and $h_{2}$ should satisfy the following conditions:
(1) $C\left(h_{1}, h_{2}\right) \geq 0$.
(2) $C\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}^{\sigma(i)}=h_{2}^{\sigma(i)}, i=1,2, \cdots, l$.

Based on Definition 2.7, we can give a cross-entropy formula of $h_{1}$ and $h_{2}$ defined as:

$$
\begin{align*}
& C_{1}\left(h_{1}, h_{2}\right) \\
& =\frac{1}{l T_{0}} \sum_{i=1}^{l}\left(\frac{\left(1+q h_{1}^{\sigma(i)}\right) \ln \left(1+q h_{1}^{\sigma(i)}\right)+\left(1+q h_{2}^{\sigma(i)}\right) \ln \left(1+q h_{2}^{\sigma(i)}\right)}{2}\right. \\
& -\frac{2+q h_{1}^{\sigma(i)}+q h_{2}^{\sigma(i)}}{2} \ln \frac{2+q h_{1}^{\sigma(i)}+q h_{2}^{\sigma(i)}}{2} \\
& \\
& +\frac{\left(1+q\left(1-h_{1}^{\sigma(l-i+1)}\right)\right) \ln \left(1+q\left(1-h_{1}^{\sigma(l-i+1)}\right)\right)+\left(1+q\left(1-h_{2}^{\sigma(l-i+1)}\right)\right) \ln \left(1+q\left(1-h_{2}^{\sigma(l-i+1)}\right)\right)}{2}  \tag{2.141}\\
& \left.-\frac{2+q\left(1-h_{1}^{\sigma(l-i+1)}+1-h_{2}^{\sigma(l-i+1)}\right)}{2} \ln \frac{2+q\left(1-h_{1}^{\sigma(l-i+1)}+1-h_{2}^{\sigma(l-i+1)}\right)}{2}\right), q>0
\end{align*}
$$

where $T_{0}=(1+q) \ln (1+q)-(2+q)(\ln (2+q)-\ln 2)$, and $q>0$.

Since

$$
\begin{equation*}
f(x)=(1+q x) \ln (1+q x), 0 \leq x \leq 1 \tag{2.142}
\end{equation*}
$$

then

$$
\begin{equation*}
f(x)_{x}^{\prime}=q \ln (1+q x)+q \geq 0, f(x)_{x}^{\prime \prime}=\frac{q^{2}}{1+q x}>0 \tag{2.143}
\end{equation*}
$$

Thus $f(x)$ is a concave-up function of $x$. Therefore, $C_{1}\left(h_{1}, h_{2}\right) \geq 0$ and $C_{1}\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}^{\sigma(i)}=h_{2}^{\sigma(i)}, i=1,2, \cdots, l$. Moreover, $C_{1}\left(h_{1}, h_{2}\right)$ degenerates to their fuzzy counterparts when $h_{1}$ and $h_{2}$ are fuzzy sets. According to Definition 2.7, $C_{1}\left(h_{1}, h_{2}\right)$ is a cross-entropy of $h_{1}$ and $h_{2}$.

Theorem 2.9 (Xu and Xia 2012a). Let $h$ be a HFE, then $E_{a}(h)=1-C_{1}\left(h, h^{c}\right)$ is an entropy formula for $h$.

Proof. $E_{a}(h)=1-C_{1}\left(h, h^{c}\right)$
$=1-\frac{2}{l_{h} T_{0}} \sum_{i=1}^{l_{h}}\left(\frac{\left(1+q h^{\sigma(i)}\right) \ln \left(1+q h^{\sigma(i)}\right)+\left(1+q\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)\right) \ln \left(1+q\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)\right)}{2}\right.$
$\left.-\frac{2+q h^{\sigma(i)}+q\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)}{2} \ln \frac{2+q h^{\sigma(i)}+q\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)}{2}\right), q>0$
where $T_{0}=(1+q) \ln (1+q)-(2+q)(\ln (2+q)-\ln 2)$, and $q>0$.
Based on Definition 2.5, the laws (1), (2) and (4) are obvious, then we only prove the law (3):
(3) If $h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)}$ for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \leq 1, i=1,2, \cdots, l$, we have

$$
\begin{equation*}
h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq 1-h_{2}^{\sigma(l-i+1)} \leq 1-h_{1}^{\sigma(l-i+1)} \tag{2.145}
\end{equation*}
$$

which means

$$
\begin{equation*}
\left|h_{1}^{\sigma(i)}+h_{1}^{\sigma(l-i+1)}-1\right| \geq\left|h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)}-1\right| \tag{2.146}
\end{equation*}
$$

Let $0 \leq x, y \leq 1$ and $d=|x-y|$, then

$$
\begin{align*}
f(x, y)= & \frac{(1+q x) \ln (1+q x)+(1+q y) \ln (1+q y)}{2} \\
& -\frac{1+q x+1+q y}{2} \ln \frac{1+q x+1+q y}{2}, q>0 \tag{2.147}
\end{align*}
$$

If $x \geq y$, then $x=d+y$, and

$$
\begin{align*}
& f(d, y)=\frac{(1+q(d+y)) \ln (1+q d+q y)+(1+q y) \ln (1+q y)}{2} \\
& -\frac{1+q(d+y)+1+q y}{2} \ln \frac{1+q(d+y)+1+q y}{2}, q>0 \tag{2.148}
\end{align*}
$$

then

$$
\begin{align*}
f(d, y)_{d}^{\prime} & =\frac{q+q \ln (1+q(y+d))+q+q \ln (1+q y)}{2} \\
& -\frac{q}{2}-\frac{q}{2} \ln \frac{1+q(y+d)+1+q y}{2} \geq 0, q>0 \tag{2.149}
\end{align*}
$$

thus $f(x, y)$ is a non-decreasing function of $|x-y|$, for $x \geq y$.
With the same reason, we can prove it is also true for $x \leq y$. Therefore, $E_{a}\left(h_{1}\right) \leq E_{a}\left(h_{2}\right)$.

Another cross-entropy formula of $h_{1}$ and $h_{2}$ can be defined as:

$$
\begin{align*}
& C_{2}\left(h_{1}, h_{2}\right)=\frac{1}{\left(1-2^{1-\lambda}\right) l} \sum_{i=1}^{l}\left(\frac{\left(h_{1}^{\sigma(i)}\right)^{\lambda}+\left(h_{2}^{\sigma(i)}\right)^{\lambda}}{2}+\frac{\left(1-h_{1}^{\sigma(l-i+1)}\right)^{\lambda}+\left(1-h_{2}^{\sigma(l-i+1)}\right)^{\lambda}}{2}\right. \\
& \left.-\left(\frac{h_{1}^{\sigma(i)}+h_{2}^{\sigma(i)}}{2}\right)^{\lambda}+\left(\frac{1-h_{1}^{\sigma(l-i+1)}+1-h_{2}^{\sigma(l-i+1)}}{2}\right)^{\lambda}\right), \lambda>1 \tag{2.150}
\end{align*}
$$

Since $g(x)=x^{\lambda}, 0 \leq x \leq 1$ and $\lambda>1$, then $g(x)_{x}^{\prime}=\lambda x^{\lambda-1}$ and $g(x)_{x}^{"}=\lambda(\lambda-1) x^{\lambda-2}>0$.

Thus, $g(x)$ is a concave-up function of $x$, and then $C_{2}\left(h_{1}, h_{2}\right) \geq 0$ and $C_{2}\left(h_{1}, h_{2}\right)=0$ if and only if $h_{1}=h_{2}$. Moreover, $C_{2}\left(h_{1}, h_{2}\right)=0$ degenerates to their fuzzy counterparts when $h_{1}$ and $h_{2}$ are fuzzy sets. According to Definition 2.7, $C_{2}\left(h_{1}, h_{2}\right)$ is a cross-entropy of $h_{1}$ and $h_{2}$.

Theorem 2.10 (Xu and Xia 2012a). Let $h$ be a HFE, then $E_{b}(h)=1-C_{2}\left(h, h^{c}\right)$ is an entropy formula for $h$.

Proof. $\quad E_{b}(h)=1-C_{2}\left(h, h^{c}\right)$
$=1-\frac{1}{\left(1-2^{1-\lambda}\right) l_{h}} \sum_{i=1}^{l_{h}}\left(\frac{\left(h^{\sigma(i)}\right)^{\lambda}+\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)^{\lambda}}{2}+\frac{\left(1-h^{\sigma(l-i+1)}\right)^{\lambda}+\left(h^{\sigma(i)}\right)^{\lambda}}{2}\right.$

$$
\left.-\left(\frac{h^{\sigma(i)}+1-h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda}+\left(\frac{1-h^{\sigma\left(l_{h}-i+1\right)}+h^{\sigma(i)}}{2}\right)^{\lambda}\right)
$$

$$
=1-\frac{2}{\left(1-2^{1-\lambda}\right) l_{h}} \sum_{i=1}^{l_{h}}\left(\frac{\left(h^{\sigma(i)}\right)^{\lambda}+\left(1-h^{\sigma\left(l_{h}-i+1\right)}\right)^{\lambda}}{2}-\left(\frac{h^{\sigma(i)}+1-h^{\sigma\left(l_{h}-i+1\right)}}{2}\right)^{\lambda}\right)
$$

$$
\begin{equation*}
\lambda>1 \tag{2.151}
\end{equation*}
$$

Based on Definition 2.5, the laws (1), (2) and (4) are obvious, then we only prove the law (3):
(3) If $h_{1} \subseteq h_{2}$, for $h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)} \leq 1$, then we have

$$
\begin{equation*}
h_{1}^{\sigma(i)} \leq h_{2}^{\sigma(i)} \leq 1-h_{2}^{\sigma(l-i+1)} \leq 1-h_{1}^{\sigma(l-i+1)} \tag{2.152}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\left|h_{1}^{\sigma(i)}+h_{1}^{\sigma(l-i+1)}-1\right| \geq\left|h_{2}^{\sigma(i)}+h_{2}^{\sigma(l-i+1)}-1\right| \tag{2.153}
\end{equation*}
$$

Let $d=|x-y|$ and

$$
\begin{equation*}
g(x, y)=\frac{x^{\lambda}+y^{\lambda}}{2}-\left(\frac{x+y}{2}\right)^{\lambda}, 0 \leq x, y \leq 1, \lambda>1 \tag{2.154}
\end{equation*}
$$

If $x \geq y$, then $x=y+d$, and

$$
\begin{gather*}
g(d, y)=\frac{(y+d)^{\lambda}+y^{\lambda}}{2}-\left(y+\frac{d}{2}\right)^{\lambda}, p>1  \tag{2.155}\\
g(d, y)_{d}^{\prime}=\frac{\lambda}{2}\left((y+d)^{\lambda-1}-\left(y+\frac{d}{2}\right)^{\lambda-1}\right) \geq 0, \lambda>1 \tag{2.156}
\end{gather*}
$$

thus $g(x, y)$ does not decrease as $|x-y|$ increases. With the same reason, we can prove it is also true for $x \leq y$. Therefore, $E_{b}\left(h_{1}\right) \leq E_{b}\left(h_{2}\right)$.

Up to now, the MADM problems have been investigated under different environments. For example, Fan and Liu (2010) proposed an approach to solving the group decision making problems with ordinal interval numbers. Xu (2007c), Xu and Chen (2008c) focused on multi-attribute group decision making with different formats of preference information on attributes. Xu and Yager (2011) developed some intuitionistic fuzzy Bonferroni means and applied them to MADM.

In the decision making process, sometimes, the information about attribute weights (Chou et al. 2008; Yeh and Chang 2009) is completely unknown because of time pressure, lack of knowledge or data, and the DM (expert)'s limited expertise about the problem domain. Some classical weight-determining methods have been developed over the last decades, including the TOPSIS method (Hwang and Yoon 1981) and the entropy method (Ye 2010), but they cannot be suitable to deal with the situation that the degrees of an alternative satisfies to an attribute are presented by several possible values which can be considered a HFE. Xu and Xia (2012a) extended the entropy method to hesitant fuzzy environment and obtained the final optimal alternative by comparing the cross-entropy measures with the ideal solutions.

Suppose that there are $n$ alternatives $A_{i}(i=1,2, \cdots, n)$ and $m$ attributes $x_{j}(j=1,2, \cdots, m)$ with the attribute weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}}$ such that $w_{j} \in[0,1], j=1,2, \cdots, m$, and $\sum_{j=1}^{m} w_{j}=1$. Suppose that a decision organization is authorized to provide all the possible degrees that the alternative $A_{i}$ satisfies the attribute $x_{j}$, denoted by a HFE $h_{i j}$.

Based on the above analysis, Xu and Xia (2012a) gave the following decision making methods:

## (Algorithm 2.1)

Step 1. The DM provides all the possible evaluations about the alternative $A_{i}$ under the attribute $x_{j}$, denoted by the HFEs $h_{i j}(i=1,2, \cdots, n$; $j=1,2, \cdots, m)$.

Step 2. If the information about the weight $w_{j}$ of the attribute $x_{j}$ is unknown completely, then we establish an exact model of entropy weights for determining the attribute weights:

$$
\begin{equation*}
w_{j}=\frac{1-E_{j}}{n-\sum_{j=1}^{m} E_{j}}, j=1,2, \cdots, m \tag{2.157}
\end{equation*}
$$

where $E_{j}=\frac{1}{n} \sum_{i=1}^{n} E\left(h_{i j}\right), \quad j=1,2, \cdots, m$.
Step 3. Let $J_{1}$ and $J_{2}$ be the sets of benefit type attributes and cost type attributes, respectively. Suppose that the hesitant fuzzy ideal solution is $h^{+}=$ $\left(h_{1}^{+}, h_{2}^{+}, \cdots, h_{n}^{+}\right)$and the hesitant fuzzy negative ideal solution is $h^{-}=\left(h_{1}^{-}, h_{2}^{-}, \cdots, h_{n}^{-}\right)$, where $h_{i}^{+}=\{1\}, h_{i}^{-}=\{0\}, i \in J_{1}$ and $h_{i}^{+}=\{0\}$, $h_{i}^{-}=\{1\}, i \in J_{2}$. Then we calculate the cross-entropy between the alternative $A_{i}$ and the positive-ideal solution or the negative-ideal solution:

$$
\begin{align*}
& C^{+}\left(A_{i}\right)=\sum_{j=1}^{m}\left(w_{j} C\left(h_{i j}, h_{j}^{+}\right)\right), i=1,2, \cdots, n  \tag{2.158}\\
& C^{-}\left(A_{i}\right)=\sum_{j=1}^{m}\left(w_{j} C\left(h_{i j}, h_{j}^{-}\right)\right), i=1,2, \cdots, n \tag{2.159}
\end{align*}
$$

Step 4. Calculate the closeness degree of the alternative $A_{i}$ to the ideal solution by using

$$
\begin{equation*}
c\left(A_{i}\right)=\frac{C^{+}\left(A_{i}\right)}{C^{+}\left(A_{i}\right)+C^{-}\left(A_{i}\right)}, i=1,2, \cdots, n \tag{2.160}
\end{equation*}
$$

Step 5. Rank the alternatives $A_{i}(i=1,2, \cdots, n)$ according to the values of $c\left(A_{i}\right)(i=1,2, \cdots, n)$ in ascending order, and the smaller the value of $c\left(A_{i}\right)$, the better the alternative $A_{i}$.

If we use the maximizing deviation method (Wang 1998) to derive the weight vector of the attributes in Step 2, and use the TOPSIS method (Hwang and Yoon 1981) to compare the alternatives in Steps 3 and 4, then the following method can be obtained:

## (Algorithm 2.2)

Step 1. See Algorithm 2.1.
Step 2. Calculate the weight vector of the attribute weight $w_{j}$ of the attribute $x_{j}$ by the maximizing deviation method:

$$
\begin{equation*}
w_{j}=\frac{\sum_{i=1}^{n} \sum_{k=1}^{n} d\left(h_{i j}, h_{k j}\right)}{\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} d\left(h_{i j}, h_{k j}\right)}, j=1,2, \cdots, m \tag{2.161}
\end{equation*}
$$

where $d\left(h_{i j}, h_{k j}\right)$ is the distance between as defined by Xu and Xia (2011c) such that for two HFEs $h_{1}$ and $h_{2}$, the distance measure between $h_{1}$ and $h_{2}$, denoted as $d\left(h_{1}, h_{2}\right)$ is defined as:

$$
\begin{equation*}
d\left(h_{1}, h_{2}\right)=\frac{1}{l} \sum_{i=1}^{l}\left|h_{1}^{\sigma(i)}-h_{2}^{\sigma(i)}\right| \tag{2.162}
\end{equation*}
$$

Step 3. Calculate the distance between the alternative $A_{i}$ and the positive-ideal solution $h^{+}=\left(h_{1}^{+}, h_{2}^{+}, \cdots, h_{m}^{+}\right)^{\mathrm{T}} \quad$ or $\quad$ the negative-ideal solution $h^{-}=\left(h_{1}^{-}, h_{2}^{-}, \cdots, h_{m}^{-}\right)^{\mathrm{T}}:$

$$
\begin{align*}
& d^{+}\left(A_{i}\right)=\sum_{j=1}^{m}\left(w_{j} d\left(h_{i j}, h_{j}^{+}\right)\right), i=1,2, \cdots, n  \tag{2.163}\\
& d^{-}\left(A_{i}\right)=\sum_{j=1}^{m}\left(w_{j} d\left(h_{i j}, h_{j}^{-}\right)\right), i=1,2, \cdots, n \tag{2.164}
\end{align*}
$$

Step 4. Calculate the closeness degree of the alternative $A_{i}$ to the ideal solution $h^{+}$by using

$$
\begin{equation*}
c\left(A_{i}\right)=\frac{d^{-}\left(A_{i}\right)}{d^{-}\left(A_{i}\right)+d^{+}\left(A_{i}\right)}, i=1,2, \cdots, n \tag{2.165}
\end{equation*}
$$

Step 5. Rank the alternatives $A_{i}(i=1,2, \cdots, n)$ according to the values of $c\left(A_{i}\right)(i=1,2, \cdots, n)$ in ascending order, and the smaller the value of $c\left(A_{i}\right)$, the better the alternative $A_{i}$.

Example 2.8 (Xu and Xia 2012a). An automotive company is desired to select the most appropriate supplier for one of the key elements in its manufacturing process (adapted from Boran et al. 2009). After pre-evaluation, four suppliers have been remained as alternatives for further evaluation. In order to evaluate alternative suppliers, four attributes are considered as: (1) $x_{1}$ : Product quality; (2) $x_{2}$ :

Relationship closeness; (3) $x_{3}$ : Delivery performance; (4) $x_{4}$ : Price, where $x_{1}$, $x_{2}$ and $x_{3}$ are the benefit-type attributes, and $x_{4}$ is the cost-type attribute.

To get the optimal alternative, the following steps are given if Algorithm 2.1 is used:

Step 1. The decision organization provides all the possible assessments of the alternative $A_{i}$ on the attribute $x_{j}$ which can be considered as a HFE $h_{i j}$ constructing the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 4}$ (see Table 2.13 (Xu and Xia 2012a)). For example, to evaluate the degrees that the alternative $A_{1}$ should satisfy the attribute $x_{1}$, some DMs in the decision organization provide 0.2 , some provide 0.4 and the others provide 0.7 , and these three parts cannot persuade each other, therefore, the degrees that the alternative $A_{1}$ satisfies the attribute $x_{1}$ can be considered a $\operatorname{HFE}\{0.2,0.4,0.7\}$.

Table 2.13. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4,0.7\}$ | $\{0.1,0.2,0.5,0.7\}$ | $\{0.2,0.3,0.5,0.7,0.8\}$ | $\{0.1,0.4,0.6\}$ |
| $A_{2}$ | $\{0.4,0.6,0.7\}$ | $\{0.1,0.2,0.4,0.6\}$ | $\{0.3,0.4,0.6,0.8,0.9\}$ | $\{0.1,0.2,0.4\}$ |
| $A_{3}$ | $\{0.2,0.3,0.6\}$ | $\{0.3,0.4,0.5,0.9\}$ | $\{0.2,0.4,0.6,0.7,0.8\}$ | $\{0.3,0.4,0.8\}$ |
| $A_{4}$ | $\{0.2,0.3,0.5\}$ | $\{0.2,0.3,0.5,0.7\}$ | $\{0.4,0.6,0.7,0.8,0.9\}$ | $\{0.1,0.2,0.7\}$ |

Step 2. Suppose that the information about the attribute weight $w_{j}$ of the attribute $x_{j}$ is unknown completely, then we calculate the entropy matrix by $E_{a}(h)(q=2)$ (see Table $2.14(\mathrm{Xu}$ and Xia 2012a)):

Table 2.14. Entropy matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.9800 | 0.9350 | 0.9920 | 0.9267 |
| $A_{2}$ | 0.9800 | 0.8750 | 0.9600 | 0.7133 |
| $A_{3}$ | 0.9200 | 0.9750 | 0.9880 | 0.9800 |
| $A_{4}$ | 0.8867 | 0.9750 | 0.8680 | 0.8533 |

and then we can obtain the weight vector:

$$
w=(0.1957,0.2013,0.1611,0.4419)^{\mathrm{T}}
$$

Step 3. Utilize $C_{1}\left(h_{1}, h_{2}\right)$ (let $q=2$ ) to calculate the cross-entropy between the alternative $A_{i}$ and the positive-ideal solution or the negative-ideal solution:

$$
\begin{aligned}
& C^{+}\left(A_{1}\right)=0.2850, C^{+}\left(A_{2}\right)=0.2040, C^{+}\left(A_{3}\right)=0.3127, C^{+}\left(A_{4}\right)=0.2648 \\
& C^{-}\left(A_{1}\right)=0.3329, C^{-}\left(A_{2}\right)=0.4275, C^{-}\left(A_{3}\right)=0.2834, C^{-}\left(A_{4}\right)=0.3747
\end{aligned}
$$

Step 4. Calculate the closeness degree of the alternative $A_{i}$ to the ideal solution:

$$
c\left(A_{1}\right)=0.4613, c\left(A_{2}\right)=0.3230, c\left(A_{3}\right)=0.5245, c\left(A_{4}\right)=0.4141
$$

Step 5. Rank the alternatives $A_{i}(i=1,2,3,4)$ according to the values of $c\left(A_{i}\right)(i=1,2,3,4)$ in ascending order:

$$
A_{2} \succ A_{4} \succ A_{1} \succ A_{3}
$$

If we use $E_{b}(h)$ and $C_{2}\left(h_{1}, h_{2}\right)$ (let $\left.p=3\right)$ in the proposed method, then the following steps are given:

Step 1. See the above.
Step 2. Utilize $E_{b}(h)$ to calculate the attribute weight vector, then

$$
w=(0.1801,0.2123,0.1615,0.4460)^{\mathrm{T}}
$$

Step 3. Utilize $C_{2}\left(h_{1}, h_{2}\right)$ to calculate the cross-entropy between the alternative $A_{i}$ and the positive-ideal solution or the negative-ideal solution, we can obtain

$$
\begin{aligned}
& C^{+}\left(A_{1}\right)=0.2916, C^{+}\left(A_{2}\right)=0.2144, C^{+}\left(A_{3}\right)=0.3169, C^{+}\left(A_{4}\right)=0.2676 \\
& C^{-}\left(A_{1}\right)=0.3393, C^{-}\left(A_{2}\right)=0.4297, C^{-}\left(A_{3}\right)=0.2929, C^{-}\left(A_{4}\right)=0.3817
\end{aligned}
$$

Step 4. Calculate the closeness degree of the alternative $A_{i}$ to the ideal solution:

$$
C\left(A_{1}\right)=0.4622, C\left(A_{2}\right)=0.3329, C\left(A_{3}\right)=0.5197, C\left(A_{4}\right)=0.4121
$$

Step 5. According to the values of $C\left(A_{i}\right)(i=1,2,3,4)$, we can get the same ranking of $A_{i}(i=1,2,3,4): A_{2} \succ A_{4} \succ A_{1} \succ A_{3}$.

If Algorithm 2.2 is used, then the following steps are given:
Step 1. See Algorithm 2.1.
Step 2. Calculate the weight vector of the attribute weight $w_{j}$ of the attribute $x_{j}$ :

$$
w=(0.2629,0.2229,0.2057,0.3086)^{\mathrm{T}}
$$

Step 3. Calculate the distance between the alternative $A_{i}$ and the positive-ideal solution or the negative-ideal solution:

$$
\begin{aligned}
& d^{-}\left(A_{1}\right)=0.4958, d^{-}\left(A_{2}\right)=0.5815, d^{-}\left(A_{3}\right)=0.4788, d^{-}\left(A_{4}\right)=0.5280 \\
& d^{+}\left(A_{1}\right)=0.5042, d^{+}\left(A_{2}\right)=0.4185, d^{+}\left(A_{3}\right)=0.5213, d^{+}\left(A_{4}\right)=0.4720
\end{aligned}
$$

Step 4. Calculate the closeness degree of the alternative $A_{i}$ to the ideal solution:

$$
c\left(A_{1}\right)=0.4958, c\left(A_{2}\right)=0.5815, c\left(A_{3}\right)=0.4788, c\left(A_{4}\right)=0.5280
$$

Step 5. Rank the alternatives $A_{i}(i=1,2,3,4)$ according to the values of $c\left(A_{i}\right)(i=1,2,3,4)$ in ascending order: $A_{2} \succ A_{4} \succ A_{1} \succ A_{3}$, which is the same as that in Algorithm 2.1.

If we use the HFWA operator (1.32) to aggregate the hesitant fuzzy information for each alternative, then according to Definition 1.2, we calculate the scores $s\left(A_{i}\right) i=1,2,3,4$ of the alternative $A_{i}(i=1,2,3,4)$, and suppose that the weight vector of alternatives, $w=(0.1957,0.2013,0.1611,0.4419)^{\mathrm{T}}$, is obtained from Algorithm 2.1, thus, we have

$$
s\left(A_{1}\right)=0.4503, s\left(A_{2}\right)=0.4683, s\left(A_{3}\right)=0.5268, s\left(A_{4}\right)=0.4793
$$

Ranking the alternatives $A_{i}(i=1,2,3,4)$ according to the values of $s\left(A_{i}\right)(i=1,2,3,4)$ in ascending order, we get the ranking result: $A_{2} \succ A_{4} \succ A_{1} \succ A_{3}$, which is just as that derived from Algorithms 2.1 and 2.2.
From the above analysis, we can find that although these three methods can get the same ranking of the alternatives, the first one focuses on the information entropy and cross entropy measures, the second on the distance measures, and both of them are suitable for dealing with the situations that the weight vector of the alternatives are unknown; The last one is only suitable for the situations that the weight vector of attributes are known. The first two methods are much simpler than the last one, because the aggregation operators in the last method need a lot of computation, which can be avoided in the first two methods. In addition, the weight vector obtained using the first method is based on the entropy method which focuses on the fuzziness of the provided information, while the weight vector obtained using the second one is based on the maximizing deviation method which focuses on the deviations among the decision information, therefore we should choose a proper method according to the practical problems.

### 2.4 Correlation Coefficients of HFSs and Their Applications to Clustering Analysis

In this section, we will introduce some formulas of the correlation coefficients for HFSs. In addition, we will apply these derived correlation coefficient formulas to do clustering analysis for hesitant fuzzy information.

Similar to the existing work (Gerstenkorn and Mańko 1991; Bustince and Burillo 1995), Chen et al. (2013a) defined the informational energy for HFSs and the corresponding correlation:

Definition 2.8 (Chen et al. 2013a). For a HFS $A=\left\{<x_{i}, h_{A}\left(x_{i}\right)>\mid x_{i} \in X\right.$, $i=1,2, \cdots, n\}$, the informational energy of the set $A$ is defined as:

$$
\begin{equation*}
\psi(A)=\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right) \tag{2.166}
\end{equation*}
$$

Definition 2.9 (Chen et al. 2013a). For two HFSs $A_{1}$ and $A_{2}$, their correlation is defined by

$$
\begin{equation*}
\rho_{1}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right) \tag{2.167}
\end{equation*}
$$

For $A_{1}, A_{2} \in \mathrm{HFSs}$, the correlation satisfies:
(1) $\rho_{1}\left(A_{1}, A_{1}\right)=\psi\left(A_{1}\right)$.
(2) $\rho_{1}\left(A_{1}, A_{2}\right)=\rho_{1}\left(A_{2}, A_{1}\right)$.

Using Definitions 2.8 and 2.9, we derive a correlation coefficient for HFSs:
Definition 2.10 (Chen et al. 2013a). The correlation coefficient between two HFSs $A_{1}$ and $A_{2}$ is given as:

$$
\begin{align*}
& c_{1}\left(A_{1}, A_{2}\right)=\frac{\rho_{1}\left(A_{1}, A_{2}\right)}{\left(\rho_{1}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{1}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}}} \\
& =\frac{\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)}{\left(\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right)^{\frac{1}{2}} \cdot\left(\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right)^{\frac{1}{2}}} \tag{2.168}
\end{align*}
$$

Theorem 2.11 (Chen et al. 2013a). The correlation coefficient between two HFSs $A_{1}$ and $A_{2}$ satisfies:
(1) $c_{1}\left(A_{1}, A_{2}\right)=c_{1}\left(A_{2}, A_{1}\right)$.
(2) $0 \leq c_{1}\left(A_{1}, A_{2}\right) \leq 1$.
(3) $c_{1}\left(A_{1}, A_{2}\right)=1$, if $A_{1}=A_{2}$.

Proof. (1) It is straightforward.
(2) The inequality $c_{1}\left(A_{1}, A_{2}\right) \geq 0$ is obvious. Below let us prove $c_{1}\left(A_{1}, A_{2}\right) \leq 1$ :

$$
\rho_{1}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)
$$

$$
\begin{align*}
& =\frac{1}{l_{x_{1}}} \sum_{j=1}^{l_{n}} h_{A_{1}}^{\sigma(j)}\left(x_{1}\right) h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)+\frac{1}{l_{x_{2}}} \sum_{j=1}^{l_{n_{2}}} h_{A_{1}}^{\sigma(j)}\left(x_{2}\right) h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)+\cdots+\frac{1}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}} h_{A_{1}}^{\sigma(j)}\left(x_{n}\right) h_{A_{2}}^{\sigma(j)}\left(x_{n}\right) \\
& =\sum_{j=1}^{l_{n}} \frac{h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)}{\sqrt{l_{x_{1}}}} \cdot \frac{h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)}{\sqrt{l_{x_{1}}}}+\sum_{j=1}^{l_{2}} \frac{h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)}{\sqrt{l_{x_{2}}}} \cdot \frac{h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)}{\sqrt{l_{x_{2}}}}+\cdots+\sum_{j=1}^{l_{x_{n}}} \frac{h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)}{\sqrt{l_{x_{n}}}} \cdot \frac{h_{2}^{\sigma(j)}\left(x_{n}\right)}{\sqrt{l_{x_{n}}}}(2.169 \tag{2.169}
\end{align*}
$$

Using the Cauchy-Schwarz inequality:

$$
\begin{equation*}
\left(x_{1} y_{1}+x_{2} y_{2}+\cdots x_{n} y_{n}\right)^{2} \leq\left(x_{1}^{2}+x_{2}^{2}+\cdots+x_{n}^{2}\right) \cdot\left(y_{1}^{2}+y_{2}^{2}+\cdots+y_{n}^{2}\right) \tag{2.170}
\end{equation*}
$$

where $\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in \mathfrak{R}^{n},\left(y_{1}, y_{2}, \cdots, y_{n}\right) \in \mathfrak{R}^{n}$, and $\mathfrak{R}$ is the set of all real numbers, then we obtain

$$
\begin{aligned}
&\left(\rho_{1}\left(A_{1}, A_{2}\right)\right)^{2} \\
& \leq\left[\sum_{j=1}^{l_{x_{1}}} \frac{1}{l_{x_{1}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\sum_{j=1}^{x_{2}} \frac{1}{l_{x_{2}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\sum_{j=1}^{l_{x_{n}}} \frac{1}{l_{x_{n}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
& \times\left[\sum_{j=1}^{l_{x_{1}}} \frac{1}{l_{x_{1}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\sum_{j=1}^{l_{x_{2}}} \frac{1}{l_{x_{2}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\sum_{j=1}^{l_{x_{n}}} \frac{1}{l_{x_{n}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
&=\left[\frac{1}{l_{x_{1}}} \sum_{j=1}^{l_{x_{1}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{1}{l_{x_{2}}} \sum_{j=1}^{l_{x_{2}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\frac{1}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
& \times\left[\frac{1}{l_{x_{1}}} \sum_{j=1}^{l_{x_{1}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{1}{l_{x_{2}}} \sum_{j=1}^{l_{x_{2}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\frac{1}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
&=\left[\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right] \cdot\left[\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right]
\end{aligned}
$$

$$
\begin{equation*}
=\rho_{1}\left(A_{1}, A_{1}\right) \cdot \rho_{1}\left(A_{2}, A_{2}\right) \tag{2.171}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\rho_{1}\left(A_{1}, A_{2}\right) \leq\left(\rho_{1}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{1}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}} \tag{2.172}
\end{equation*}
$$

So $0 \leq c_{1}\left(A_{1}, A_{2}\right) \leq 1$.
(3) $A_{1}=A_{2} \Rightarrow h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)=h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\left(x_{i} \in X\right) \Rightarrow c_{1}\left(A_{1}, A_{2}\right)=1$.

Example 2.9 (Chen et al. 2013a). Let $A_{1}$ and $A_{2}$ be two HFSs in $X=\left\{x_{1}, x_{2}, x_{3}\right\}$, and

$$
\begin{aligned}
& A_{1}=\left\{\left\langle x_{1},\{0.7,0.5\}\right\rangle,\left\langle x_{2},\{0.9,0.8,0.6\}\right\rangle,\left\langle x_{3},\{0.5,0.4,0.2\}\right\rangle\right\} \\
& A_{2}=\left\{\left\langle x_{1},\{0.4,0.2\}\right\rangle,\left\langle x_{2},\{0.8,0.5,0.4\}\right\rangle,\left\langle x_{3},\{0.7,0.6,0.3\}\right\rangle\right\}
\end{aligned}
$$

Then we calculate

$$
\begin{aligned}
& \rho_{1}\left(A_{1}, A_{1}\right)=\psi\left(A_{1}\right)=\sum_{i=1}^{3}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right) \\
& =\frac{1}{2} \sum_{j=1}^{2}\left(h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{1}{3} \sum_{j=1}^{3}\left(h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\frac{1}{3} \sum_{j=1}^{3}\left(h_{A_{1}}^{\sigma(j)}\left(x_{3}\right)\right)^{2} \\
& =\frac{1}{2}\left(0.7^{2}+0.5^{2}\right)+\frac{1}{3}\left(0.9^{2}+0.8^{2}+0.6^{2}\right)+\frac{1}{3}\left(0.5^{2}+0.4^{2}+0.2^{2}\right) \\
& =1.1233
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\rho_{1}\left(A_{2}, A_{2}\right) & =\psi\left(A_{2}\right)=\sum_{i=1}^{3}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right) \\
& =\frac{1}{2} \sum_{j=1}^{2}\left(h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{1}{3} \sum_{j=1}^{3}\left(h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\frac{1}{3} \sum_{j=1}^{3}\left(h_{A_{2}}^{\sigma(j)}\left(x_{3}\right)\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{2}\left(0.4^{2}+0.2^{2}\right)+\frac{1}{3}\left(0.8^{2}+0.5^{2}+0.4^{2}\right)+\frac{1}{3}\left(0.7^{2}+0.6^{2}+0.3^{2}\right) \\
& =0.7633
\end{aligned}
$$

$$
\begin{aligned}
& \rho_{1}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{3}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right) \\
& \quad=\frac{1}{2} \sum_{j=1}^{2} h_{A_{1}}^{\sigma(j)}\left(x_{1}\right) h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)+\frac{1}{3} \sum_{j=1}^{3} h_{A_{1}}^{\sigma(j)}\left(x_{2}\right) h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)+\frac{1}{3} \sum_{j=1}^{3} h_{A_{1}}^{\sigma(j)}\left(x_{3}\right) h_{A_{2}}^{\sigma(j)}\left(x_{3}\right) \\
& \quad=\frac{1}{2}(0.7 \times 0.4+0.5 \times 0.2)+\frac{1}{3}(0.9 \times 0.8+0.8 \times 0.5+0.6 \times 0.4)+\frac{1}{3}(0.5 \times 0.7 \\
& \quad+0.4 \times 0.6+0.2 \times 0.3)=0.86
\end{aligned}
$$

and then, we have

$$
\begin{aligned}
c_{1}\left(A_{1}, A_{2}\right) & =\frac{\rho_{1}\left(A_{1}, A_{2}\right)}{\left(\rho_{1}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{1}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}}} \\
& =\frac{0.86}{\sqrt{1.1233} \cdot \sqrt{0.7633}}=0.9288
\end{aligned}
$$

Obviously, $0<c_{1}\left(A_{1}, A_{2}\right)<1$.
In what follows, we give a new formula of calculating the correlation coefficient of HFSs, which is similar to that used in IFSs (Xu et al. 2008):

Definition 2.11. (Chen et al. 2013a) For two HFSs $A_{1}$ and $A_{2}$, their correlation coefficient is defined by

$$
\begin{align*}
& c_{2}\left(A_{1}, A_{2}\right)=\frac{\rho_{1}\left(A_{1}, A_{2}\right)}{\max \left\{\rho_{1}\left(A_{1}, A_{1}\right), \rho_{1}\left(A_{2}, A_{2}\right)\right\}} \\
&=\frac{\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)}{\max \left\{\sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right), \sum_{i=1}^{n}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right\}} \tag{2.173}
\end{align*}
$$

Theorem 2.12 (Chen et al. 2013a). The correlation coefficient of two HFSs $A_{1}$ and $A_{2}, c_{2}\left(A_{1}, A_{2}\right)$, follows the same properties listed in Theorem 2.11.

Proof. The process to prove the properties (1) and (3) is analogous to that in Theorem 2.11, we do not repeat it here.
(2) $c_{2}\left(A_{1}, A_{2}\right) \geq 0$ is obvious. We now only prove $c_{2}\left(A_{1}, A_{2}\right) \leq 1$.

Based on the proof process of Theorem 2.11, we have

$$
\begin{equation*}
\rho_{1}\left(A_{1}, A_{2}\right) \leq\left(\rho_{1}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{1}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}} \tag{2.174}
\end{equation*}
$$

and then

$$
\begin{equation*}
\rho_{1}\left(A_{1}, A_{2}\right) \leq \max \left\{\rho_{1}\left(A_{1}, A_{1}\right), \rho_{1}\left(A_{2}, A_{2}\right)\right\} \tag{2.175}
\end{equation*}
$$

Thus, $c_{2}\left(A_{1}, A_{2}\right) \leq 1$.
In practical applications, the elements $x_{i}(i=1,2, \cdots, n)$ in the universe $X$ have different weights. Let $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ be the weight vector of $x_{i}(i=1,2, \cdots, n)$ with $w_{i} \geq 0, i=1,2, \cdots, n$ and $\sum_{i=1}^{n} w_{i}=1$, we further extend the correlation coefficient formulas $c_{1}\left(A_{1}, A_{2}\right)$ and $c_{2}\left(A_{1}, A_{2}\right)$ as:

$$
\begin{align*}
& c_{3}\left(A_{1}, A_{2}\right)=\frac{\rho_{2}\left(A_{1}, A_{2}\right)}{\left(\rho_{2}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{2}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}}} \\
& =\frac{\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)}{\left(\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right)^{\frac{1}{2}} \cdot\left(\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right)^{\frac{1}{2}}} \tag{2.176}
\end{align*}
$$

$$
\begin{align*}
& c_{4}\left(A_{1}, A_{2}\right)=\frac{\rho_{2}\left(A_{1}, A_{2}\right)}{\max \left\{\rho_{2}\left(A_{1}, A_{1}\right), \rho_{2}\left(A_{2}, A_{2}\right)\right\}} \\
& =\frac{\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)}{\max \left\{\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right), \sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right\}} \tag{2.177}
\end{align*}
$$

It can be seen that if $w=\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{\mathrm{T}}$, then $c_{3}\left(A_{1}, A_{2}\right)$ and $c_{4}\left(A_{1}, A_{2}\right)$ reduce to $c_{1}\left(A_{1}, A_{2}\right)$ and $c_{2}\left(A_{1}, A_{2}\right)$, respectively. Note that both $c_{3}\left(A_{1}, A_{2}\right)$ and $c_{4}\left(A_{1}, A_{2}\right)$ also satisfy three properties of Theorem 2.11.

Theorem 2.13 (Chen et al. 2013a). Let $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ be the weight vector of $x_{i}(i=1,2, \cdots, n)$ with $w_{i} \geq 0, i=1,2, \cdots, n$ and $\sum_{i=1}^{n} w_{i}=1$, the correlation coefficient $c_{3}\left(A_{1}, A_{2}\right)$ between two HFSs $A_{1}$ and $A_{2}$, which takes into account the weights, satisfies:
(1) $c_{3}\left(A_{1}, A_{2}\right)=c_{3}\left(A_{2}, A_{1}\right)$.
(2) $0 \leq c_{3}\left(A_{1}, A_{2}\right) \leq 1$.
(3) $c_{3}\left(A_{1}, A_{2}\right)=1$, if $A_{1}=A_{2}$.

Proof. (1) It is straightforward.
(2) $c_{3}\left(A_{1}, A_{2}\right) \geq 0$ is obvious. Below we prove $c_{3}\left(A_{1}, A_{2}\right) \leq 1$. Since

$$
\begin{aligned}
& \rho_{2}\left(A_{1}, A_{2}\right)=\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}} h_{A_{1}}^{\sigma(j)}\left(x_{i}\right) h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right) \\
& =\frac{w_{1}}{l_{x_{1}}} \sum_{j=1}^{l_{x_{1}}} h_{A_{1}}^{\sigma(j)}\left(x_{1}\right) h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)+\frac{w_{2}}{l_{x_{2}}} \sum_{j=1}^{l_{x_{2}}} h_{A_{1}}^{\sigma(j)}\left(x_{2}\right) h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\cdots+\frac{w_{n}}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}} h_{A_{1}}^{\sigma(j)}\left(x_{n}\right) h_{A_{2}}^{\sigma(j)}\left(x_{n}\right) \\
= & \sum_{j=1}^{l_{x_{n}}} \frac{\sqrt{w_{1}} \cdot h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)}{\sqrt{l_{x_{1}}}} \cdot \frac{\sqrt{w_{1}} \cdot h_{A_{2}}^{\sigma(j)}}{\sqrt{l_{x_{1}}}}\left(x_{1}\right)  \tag{2.178}\\
& \cdots+\sum_{j=1}^{l_{m n}} \frac{\sqrt{w_{n}} \cdot h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)}{\sqrt{l_{x_{n}}}} \cdot \frac{\sqrt{w_{n}} \cdot h_{A_{2}}^{\sigma(j)}\left(x_{n}\right)}{\sqrt{l_{x_{n}}}}
\end{align*}
$$

and by using the Cauchy-Schwarz inequality, we obtain

$$
\begin{align*}
& \left(\rho_{2}\left(A_{1}, A_{2}\right)\right)^{2} \\
& \leq\left[\sum_{j=1}^{l_{x_{1}}} \frac{w_{1}}{l_{x_{1}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\sum_{j=1}^{l_{x_{2}}} \frac{w_{2}}{l_{x_{2}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\sum_{j=1}^{l_{x_{n}}} \frac{w_{n}}{l_{x_{n}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
& \\
& \times\left[\sum_{j=1}^{l_{x_{1}}} \frac{w_{1}}{l_{x_{1}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\sum_{j=1}^{l_{x_{2}}} \frac{w_{2}}{l_{x_{2}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\sum_{j=1}^{l_{x_{n}}} \frac{w_{n}}{l_{x_{n}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
& =\left[\frac{w_{1}}{l_{x_{1}}} \sum_{j=1}^{l_{x_{1}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{w_{2}}{l_{x_{2}}} \sum_{j=1}^{l_{x_{2}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\frac{w_{n}}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
& \\
& \times\left[\frac{w_{1}}{l_{x_{1}}} \sum_{j=1}^{l_{x_{1}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{1}\right)\right)^{2}+\frac{w_{2}}{l_{x_{2}}} \sum_{j=1}^{l_{x_{2}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{2}\right)\right)^{2}+\cdots+\frac{w_{n}}{l_{x_{n}}} \sum_{j=1}^{l_{x_{n}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{n}\right)\right)^{2}\right] \\
&  \tag{2.179}\\
& =\left[\sum_{i=1}^{n}\left(\frac{w_{i}}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right] \cdot\left[\sum_{i=1}^{n}\left(\frac{w_{i}}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right] \\
& \\
& =\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right] \cdot\left[\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{x_{i}}} \sum_{j=1}^{l_{x_{i}}}\left(h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\right)^{2}\right)\right]\right.
\end{align*}
$$

Thus

$$
\begin{equation*}
\rho_{2}\left(A_{1}, A_{2}\right) \leq\left(\rho_{2}\left(A_{1}, A_{1}\right)\right)^{\frac{1}{2}} \cdot\left(\rho_{2}\left(A_{2}, A_{2}\right)\right)^{\frac{1}{2}} \tag{2.180}
\end{equation*}
$$

That is, $c_{3}\left(A_{1}, A_{2}\right) \leq 1$.
(3) $A_{1}=A_{2} \Rightarrow h_{A_{1}}^{\sigma(j)}\left(x_{i}\right)=h_{A_{2}}^{\sigma(j)}\left(x_{i}\right)\left(x_{i} \in X\right) \Rightarrow c_{3}\left(A_{1}, A_{2}\right)=1$.

Theorem 2.14 (Chen et al. 2013a). The correlation coefficient of two HFSs $A_{1}$ and $A_{2}$ defined in $c_{2}\left(A_{1}, A_{2}\right)$, which accounts for the weights, $c_{4}\left(A_{1}, A_{2}\right)$, satisfies the same properties as those in Theorem 2.13.

Since the process to prove these properties is analogous to that in Theorem 2.12, we do not repeat it here.

Example 2.10 (Chen et al. 2013a). Let $A_{1}, A_{2}$, and $A_{3}$ be three HFSs in $X=\left\{x_{1}, x_{2}, x_{3}\right\}, w=(0.3,0.3,0.4)^{\mathrm{T}}$ the weight vector of $x_{i}(i=1,2,3)$, and

$$
\begin{aligned}
& A_{1}=\left\{\left\langle x_{1},\{0.9,0.8,0.5\}\right\rangle,\left\langle x_{2},\{0.2,0.1\}\right\rangle,\left\langle x_{3},\{0.5,0.3,0.2,0.1\}\right\rangle\right\} \\
& A_{2}=\left\{\left\langle x_{1},\{0.7,0.5,0.4\}\right\rangle,\left\langle x_{2},\{0.5,0.3\}\right\rangle,\left\langle x_{3},\{0.6,0.4,0.3,0.1\}\right\rangle\right\} \\
& A_{3}=\left\{\left\langle x_{1},\{0.3,0.2,0.1\}\right\rangle,\left\langle x_{2},\{0.3,0.2\}\right\rangle,\left\langle x_{3},\{0.8,0.7,0.5,0.4\}\right\rangle\right\}
\end{aligned}
$$

Then we can obtain

$$
c_{3}\left(A_{1}, A_{2}\right)=0.9135, c_{3}\left(A_{1}, A_{3}\right)=0.6700, c_{3}\left(A_{2}, A_{3}\right)=0.8278
$$

Obviously, $c_{3}\left(A_{1}, A_{2}\right)>c_{3}\left(A_{2}, A_{3}\right)>c_{3}\left(A_{1}, A_{3}\right)$.
Based on the intuitionistic fuzzy clustering algorithm (Xu et al. 2008), and the correlation coefficient formulas developed previously for HFSs, Chen et al. (2013a) developed an algorithm to do clustering under hesitant fuzzy environments. Before doing this, some concepts are introduced firstly:

Definition 2.12 (Chen et al. 2013a). Let $A_{i}(i=1,2, \cdots, n)$ be $n$ HFSs, and $C=\left(c_{i j}\right)_{n \times n}$ a correlation matrix, where $c_{i j}=c\left(A_{i}, A_{j}\right)$ denotes the correlation coefficient of two HFSs $A_{i}$ and $A_{j}$ and satisfies:
(1) $0 \leq c_{i j} \leq 1, i, j=1,2, \cdots, n$.
(2) $c_{i i}=1, i=1,2, \cdots, n$.
(3) $c_{i j}=c_{j i}, i, j=1,2, \cdots, n$.

Definition 2.13 ( Xu et al. 2008). Let $C=\left(c_{i j}\right)_{n \times n}$ be a correlation matrix, if $C^{2}=C \circ C=\left(\bar{c}_{i j}\right)_{n \times n}$, then $C^{2}$ is called a composition matrix of $C$, where

$$
\begin{equation*}
\bar{c}_{i j}=\max _{k}\left\{\min \left\{c_{i k}, c_{k j}\right\}\right\}, \quad i, j=1,2, \cdots, n \tag{2.181}
\end{equation*}
$$

Theorem 2.15 ( Xu et al. 2008). Let $C=\left(c_{i j}\right)_{n \times n}$ be a correlation matrix. Then the composition matrix $C^{2}=C \circ C=\left(\bar{c}_{i j}\right)_{n \times n}$ is also a correlation matrix.

Theorem 2.16 ( Xu et al. 2008). Let $C$ be a correlation matrix. Then for any nonnegative integers $m_{1}$ and $m_{2}$, the composition matrix $C^{m_{1}+m_{2}}$ derived from $C^{m_{1}+m_{2}}=C^{m_{1}} \circ C^{m_{2}}$ is still a correlation matrix.

Definition 2.14 ( Xu et al. 2008). Let $C=\left(c_{i j}\right)_{n \times n}$ be a correlation matrix, if $C^{2} \subseteq C$, i.e.,

$$
\begin{equation*}
\max _{k}\left\{\min \left\{c_{i k}, c_{k j}\right\}\right\} \leq c_{i j}, \quad i, j=1,2, \cdots, n \tag{2.182}
\end{equation*}
$$

then $C$ is called an equivalent correlation matrix.
Theorem 2.17 (Wang 1983; Xu et al. 2008). Let $C=\left(c_{i j}\right)_{n \times n}$ be a correlation matrix. Then after the finite times of compositions: $C \rightarrow C^{2} \rightarrow C^{4} \rightarrow \cdots \rightarrow C^{2^{k}} \rightarrow \cdots$, there must exist a positive integer $k$ such that $C^{2^{k}}=C^{2^{(k+1)}}$ and $C^{2^{k}}$ is also an equivalent correlation matrix.

Definition 2.15 ( Xu et al. 2008). Let $C=\left(c_{i j}\right)_{n \times n}$ be an equivalent correlation matrix. Then we call $C_{\lambda_{0}}=\left({ }_{\lambda_{0}} c_{i j}\right)_{n \times n}$ the $\lambda_{0}$-cutting matrix of $C$, where

$$
{ }_{\lambda 0} c_{i j}=\left\{\begin{array}{ll}
0, & \text { if } c_{i j}<\lambda_{0},  \tag{2.183}\\
1, & \text { if } c_{i j} \geq \lambda_{0},
\end{array} \quad i, j=1,2, \cdots, n\right.
$$

and $\lambda_{0}$ is the confidence level with $\lambda_{0} \in[0,1]$.
Chen et al. (2013a) proposed an algorithm for clustering HFSs as follows:

## (Algorithm 2.3)

Step 1. Let $\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a set of HFSs on $X=\left\{x_{1}, x_{2}, \cdots x_{m}\right\}$. We can calculate the correlation coefficients of the HFSs, and then construct a correlation matrix $C=\left(c_{i j}\right)_{n \times n}$, where $c_{i j}=c\left(A_{i}, A_{j}\right)$.

Step 2. Check whether $C=\left(c_{i j}\right)_{n \times n}$, is an equivalent correlation matrix, i.e., check whether it satisfies $C^{2} \subseteq C$, where

$$
\begin{equation*}
C^{2}=C \circ C=\left(\bar{c}_{i j}\right)_{n \times n}, \bar{c}_{i j}=\max _{k}\left\{\min \left\{c_{i k}, c_{k_{j}}\right\}\right\}, \quad i, j=1,2, \cdots, n \tag{2.184}
\end{equation*}
$$

If it does not hold, then we construct the equivalent correlation matrix $C^{2^{k}}$ :

$$
\begin{equation*}
C \rightarrow C^{2} \rightarrow C^{4} \rightarrow \cdots \rightarrow C^{2^{k}} \rightarrow \cdots, \text { until } C^{2^{k}}=C^{2^{(k+1)}} \tag{2.185}
\end{equation*}
$$

Step 3. For a confidence level $\lambda_{0}$, we construct a $\lambda_{0}$-cutting matrix $C_{\lambda_{0}}=$ $\left(\lambda_{0} c_{i j}\right)_{n \times n}$ in order to classify the HFSs $A_{i}(i=1,2, \cdots, n)$. If all elements of the $i$ th line (column) in $C_{\lambda_{0}}$ are the same as the corresponding elements of the $j$ th line (column) in $C_{\lambda_{0}}$, then the HFSs $A_{i}$ and $A_{j}$ are of the same type. By means of this principle, we can classify all these $n$ HFSs $A_{i}(i=1,2, \cdots, n)$.

Below two real examples are employed to illustrate the need of the clustering algorithm based on HFSs:

Example 2.11 (Chen et al. 2013a). Software evaluation and classification is an increasingly important problem in any sector of human activity. Industrial production, service provisioning and business administration heavily depend on software which is more and more complex and expensive (Stamelos and Tsoukiàs 2003). A CASE tool to support the production of software in a CIM environment
has to be selected from the ones offered on the market. CIM software typically has responsibility for production planning, production control and monitoring (Morisio and Tsoukiàs 1997).

To better evaluate different types of CIM softwares $A_{i}(i=1,2, \ldots, 7)$ on the market, we perform clustering for them according to four attributes: (1) $x_{1}$ :

Functionality; (2) $x_{2}$ : Usability; (3) $x_{3}$ : Portability; (4) $x_{4}$ : Maturity. Given the DMs (experts) who make such an evaluation have different backgrounds and levels of knowledge, skills, experience and personality, etc., this could lead to a difference in the evaluation information. To clearly reflect the differences of the opinions of different DMs, the data of evaluation information are represented by the HFSs and listed in Table 2.15 (Chen et al. 2013a).

Table 2.15. Hesitant fuzzy information

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.9,0.85,0.8\}$ | $\{0.8,0.75,0.7\}$ | $\{0.8,0.65\}$ | $\{0.35,0.3\}$ |
| $A_{2}$ | $\{0.9,0.85\}$ | $\{0.8,0.7,0.6\}$ | $\{0.2\}$ | $\{0.15\}$ |
| $A_{3}$ | $\{0.4,0.3,0.2\}$ | $\{0.5,0.4\}$ | $\{1.0,0.9\}$ | $\{0.65,0.5,0.45\}$ |
| $A_{4}$ | $\{1.0,0.95,0.8\}$ | $\{0.2,0.15,0.1\}$ | $\{0.3,0.2\}$ | $\{0.8,0.7,0.6\}$ |
| $A_{5}$ | $\{0.5,0.4,0.35\}$ | $\{1.0,0.9,0.7\}$ | $\{0.4\}$ | $\{0.35,0.3,0.2\}$ |
| $A_{6}$ | $\{0.7,0.6,0.5\}$ | $\{0.9,0.8\}$ | $\{0.6,0.4\}$ | $\{0.2,0.1\}$ |
| $A_{7}$ | $\{1,0.8\}$ | $\{0.35,0.2,0.15\}$ | $\{0.2,0.1\}$ | $\{0.85,0.7\}$ |

Step 1. Calculate the correlation coefficients of the CIM softwares $A_{i}(i=$ $1,2, \cdots, 7)$ by using $c_{3}\left(A_{1}, A_{2}\right)$ with the weighting vector $w=(0.35,0.30,0.15,0.2)^{\mathrm{T}}$, and let $l_{x_{j}}=\max \left\{l\left(h_{A_{i}}\left(x_{j}\right)\right)\right\}, i=1,2, \cdots, 7$. Then the derived correlation matrix is:

$$
C=\left(\begin{array}{lllllll}
1.0000 & 0.9531 & 0.8461 & 0.8192 & 0.9182 & 0.9686 & 0.8233 \\
0.9531 & 1.0000 & 0.6573 & 0.8128 & 0.8861 & 0.9418 & 0.8292 \\
0.8461 & 0.6573 & 1.0000 & 0.6722 & 0.8041 & 0.7939 & 0.6732 \\
0.8192 & 0.8128 & 0.6722 & 1.0000 & 0.6243 & 0.6855 & 0.9906 \\
0.9182 & 0.8861 & 0.8041 & 0.6243 & 1.0000 & 0.9702 & 0.6671 \\
0.9686 & 0.9418 & 0.7639 & 0.6855 & 0.9702 & 1.0000 & 0.7074 \\
0.8233 & 0.8292 & 0.6732 & 0.9906 & 0.6671 & 0.7074 & 1.0000
\end{array}\right)
$$

Step 2. Construct the equivalent correlation matrix and calculate
$C^{2}=C \circ C=\left(\begin{array}{lllllll}1.0000 & 0.9531 & 0.8461 & 0.8233 & 0.9686 & 0.9686 & 0.8292 \\ 0.9531 & 1.0000 & 0.8461 & 0.8292 & 0.9418 & 0.9531 & 0.8292 \\ 0.8461 & 0.8461 & 1.0000 & 0.8192 & 0.8461 & 0.8461 & 0.8233 \\ 0.8233 & 0.8292 & 0.8192 & 1.0000 & 0.8192 & 0.8192 & 0.9906 \\ 0.9686 & 0.9418 & 0.8461 & 0.8192 & 1.0000 & 0.9702 & 0.8292 \\ 0.9686 & 0.9531 & 0.8461 & 0.8192 & 0.9702 & 1.0000 & 0.8292 \\ 0.8292 & 0.8292 & 0.8233 & 0.9906 & 0.8292 & 0.8292 & 1.0000\end{array}\right)$

It can be seen that $C^{2} \subseteq C$ does not hold. That is to say, the correlation matrix $C$ is not an equivalent correlation matrix. So, we further calculate
$C^{4}=C^{2} \circ C^{2}=\left(\begin{array}{lllllll}1.0000 & 0.9531 & 0.8461 & 0.8292 & 0.9686 & 0.9686 & 0.8292 \\ 0.9531 & 1.0000 & 0.8461 & 0.8292 & 0.9531 & 0.9531 & 0.8292 \\ 0.8461 & 0.8461 & 1.0000 & 0.8292 & 0.8461 & 0.8461 & 0.8292 \\ 0.8292 & 0.8292 & 0.8292 & 1.0000 & 0.8292 & 0.8292 & 0.9906 \\ 0.9686 & 0.9531 & 0.8461 & 0.8292 & 1.0000 & 0.9702 & 0.8292 \\ 0.9686 & 0.9531 & 0.8461 & 0.8292 & 0.9702 & 1.0000 & 0.8292 \\ 0.8292 & 0.8292 & 0.8292 & 0.9906 & 0.8292 & 0.8292 & 1.0000\end{array}\right)$
and
$C^{8}=C^{4} \circ C^{4}=\left(\begin{array}{lllllll}1.0000 & 0.9531 & 0.8461 & 0.8292 & 0.9686 & 0.9686 & 0.8292 \\ 0.9531 & 1.0000 & 0.8461 & 0.8292 & 0.9531 & 0.9531 & 0.8292 \\ 0.8461 & 0.8461 & 1.0000 & 0.8292 & 0.8461 & 0.8461 & 0.8292 \\ 0.8292 & 0.8292 & 0.8292 & 1.0000 & 0.8292 & 0.8292 & 0.9906 \\ 0.9686 & 0.9531 & 0.8461 & 0.8292 & 1.0000 & 0.9702 & 0.8292 \\ 0.9686 & 0.9531 & 0.8461 & 0.8292 & 0.9702 & 1.0000 & 0.8292 \\ 0.8292 & 0.8292 & 0.8292 & 0.9906 & 0.8292 & 0.8292 & 1.0000\end{array}\right)$

$$
=C^{4}
$$

Hence, $C^{4}$ is an equivalent correlation matrix.

Step 3. For a confidence level $\lambda_{0}$, to do clustering for the CIM softwares $A_{i}(i=1,2, \cdots, 7)$, we construct a $\lambda_{0}$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} c_{i j}\right)_{7 \times 7}$ and based on which, we get all possible classifications of $A_{i}(i=1,2, \cdots, 7)$ :
(1) If $0 \leq \lambda_{0} \leq 0.8292$, then $A_{i}(i=1,2, \cdots, 7)$ are of the same type:

$$
\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}\right\}
$$

(2) If $0.8292<\lambda_{0} \leq 0.8461$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into two types:

$$
\left\{A_{1}, A_{2}, A_{3}, A_{5}, A_{6}\right\},\left\{A_{4}, A_{7}\right\}
$$

(3) If $0.8461<\lambda_{0} \leq 0.9531$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into three types:

$$
\left\{A_{1}, A_{2}, A_{5}, A_{6}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{7}\right\}
$$

(4) If $0.9531<\lambda_{0} \leq 0.9686$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into four types:

$$
\left\{A_{1}, A_{5}, A_{6}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{7}\right\}
$$

(5) If $0.9686<\lambda_{0} \leq 0.9702$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into five types:

$$
\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{5}, A_{6}\right\},\left\{A_{4}, A_{7}\right\}
$$

(6) If $0.9702<\lambda_{0} \leq 0.9906$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into six types:

$$
\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{4}, A_{7}\right\}
$$

(7) If $0.9906<\lambda_{0} \leq 1$, then $A_{i}(i=1,2, \cdots, 7)$ are classified into seven types:

$$
\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\}
$$

Example 2.12 (Chen et al. 2013a). The assessment of business failure risk, i.e., the assessment of firm performance and the prediction of failure events has drawn the attention of many researchers in recent years (Iopounidis 1987; Dimitras et al. 1995).

For this purpose, 10 firms $A_{i}(i=1,2, \ldots, 10)$ evaluated on five attributes $\left(x_{1}\right.$ : managers work experience, $x_{2}$ : profitability, $x_{3}$ : operating capacity, $x_{4}$ : debt-paying ability, and $x_{5}$ : market competition) will be classified according to their risk of failure. In order to better make the assessment, several risk evaluation organizations are requested. The normalized evaluation data, represented by HFSs, are displayed in Table 2.16 (Chen et al. 2013a).

Table 2.16. The evaluation information for the 5 attributes of 10 firms

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.3,0.4,0.5\}$ | $\{0.4,0.5\}$ | $\{0.8\}$ | $\{0.5\}$ | $\{0.2,0.3\}$ |
| $A_{2}$ | $\{0.4,0.6\}$ | $\{0.6,0.8\}$ | $\{0.2,0.3\}$ | $\{0.3,0.4\}$ | $\{0.6,0.7,0.9\}$ |
| $A_{3}$ | $\{0.5,0.7\}$ | $\{0.9\}$ | $\{0.3,0.4\}$ | $\{0.3\}$ | $\{0.8,0.9\}$ |
| $A_{4}$ | $\{0.3,0.4,0.5\}$ | $\{0.8,0.9\}$ | $\{0.7,0.9\}$ | $\{0.1,0.2\}$ | $\{0.9,1.0\}$ |
| $A_{5}$ | $\{0.8,1.0\}$ | $\{0.8,1.0\}$ | $\{0.4,0.6\}$ | $\{0.8\}$ | $\{0.7,0.8\}$ |
| $A_{6}$ | $\{0.4,0.5,0.6\}$ | $\{0.2,0.3\}$ | $\{0.9,1.0\}$ | $\{0.5\}$ | $\{0.3,0.4,0.5\}$ |
| $A_{7}$ | $\{0.6\}$ | $\{0.7,0.9\}$ | $\{0.8\}$ | $\{0.3,0.4\}$ | $\{0.4,0.7\}$ |
| $A_{8}$ | $\{0.9,1.0\}$ | $\{0.7,0.8\}$ | $\{0.4,0.5\}$ | $\{0.5,0.6\}$ | $\{0.7\}$ |
| $A_{9}$ | $\{0.4,0.6\}$ | $\{1.0\}$ | $\{0.6,0.7\}$ | $\{0.2,0.3\}$ | $\{0.9,1.0\}$ |
| $A_{10}$ | $\{0.9\}$ | $\{0.6,0.7\}$ | $\{0.5,0.8\}$ | $\{1.0\}$ | $\{0.7,0.8,0.9\}$ |

Step 1. Calculate the correlation coefficients of the HFSs $A_{i}(i=1,2, \cdots, 10)$ by using $C_{3}\left(A_{i}, A_{j}\right)$ with the weighting vector $w=(0.15,0.3,0.2,0.25,0.1)^{\mathrm{T}}$, and let $l_{x_{j}}=\max \left\{l\left(h_{A_{i}}\left(x_{j}\right)\right)\right\}, i=1,2, \cdots, 10$. Then the correlation matrix derived is:
$C=\left(\begin{array}{llllllllll}1.0000 & 0.7984 & 0.6583 & 0.6635 & 0.5964 & 0.9104 & 0.7572 & 0.6761 & 0.6147 & 0.5983 \\ 0.7984 & 1.0000 & 0.8200 & 0.7139 & 0.6459 & 0.6666 & 0.7411 & 0.7458 & 0.7052 & 0.5855 \\ 0.6583 & 0.8200 & 1.0000 & 0.8813 & 0.7593 & 0.6082 & 0.8997 & 0.8872 & 0.8683 & 0.6757 \\ 0.6635 & 0.7139 & 0.8813 & 1.0000 & 0.7423 & 0.6542 & 0.9238 & 0.8743 & 0.9306 & 0.6742 \\ 0.5964 & 0.6459 & 0.7593 & 0.7423 & 1.0000 & 0.5761 & 0.7737 & 0.8520 & 0.8253 & 0.9515 \\ 0.9104 & 0.6666 & 0.6082 & 0.6542 & 0.5761 & 1.0000 & 0.7427 & 0.6647 & 0.5816 & 0.6124 \\ 0.7572 & 0.7411 & 0.8997 & 0.9238 & 0.7737 & 0.7427 & 1.0000 & 0.9025 & 0.8723 & 0.7217 \\ 0.6761 & 0.7458 & 0.8872 & 0.8743 & 0.8520 & 0.6647 & 0.9025 & 1.0000 & 0.8617 & 0.8067 \\ 0.6147 & 0.7052 & 0.8683 & 0.9306 & 0.8253 & 0.5816 & 0.8723 & 0.8617 & 1.0000 & 0.7377 \\ 0.5983 & 0.5855 & 0.6757 & 0.6742 & 0.9515 & 0.6124 & 0.7217 & 0.8067 & 0.7377 & 1.0000\end{array}\right)$

Step 2. Construct the equivalent correlation matrix and obtain

$$
C^{16}=C^{8} \circ C^{8}
$$

$=\left(\begin{array}{llllllllll}1.0000 & 0.7984 & 0.7984 & 0.7984 & 0.7984 & 0.9104 & 0.7984 & 0.7984 & 0.7984 & 0.7984 \\ 0.7984 & 1.0000 & 0.8200 & 0.8200 & 0.8200 & 0.7984 & 0.8200 & 0.8200 & 0.8200 & 0.8200 \\ 0.7984 & 0.8200 & 1.0000 & 0.8997 & 0.8520 & 0.7984 & 0.8997 & 0.8997 & 0.8997 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 1.0000 & 0.8520 & 0.7984 & 0.9238 & 0.9025 & 0.9306 & 0.8520 \\ 0.7984 & 0.8200 & 0.8520 & 0.8520 & 1.0000 & 0.7984 & 0.8520 & 0.8520 & 0.8520 & 0.9515 \\ 0.9104 & 0.7984 & 0.7984 & 0.7984 & 0.7984 & 1.0000 & 0.7984 & 0.7984 & 0.7984 & 0.7984 \\ 0.7984 & 0.8200 & 0.8997 & 0.9238 & 0.8520 & 0.7984 & 1.0000 & 0.9025 & 0.9238 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 0.9025 & 0.8520 & 0.7984 & 0.9025 & 1.0000 & 0.9025 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 0.9306 & 0.8520 & 0.7984 & 0.9238 & 0.9025 & 1.0000 & 0.8520 \\ 0.7984 & 0.8200 & 0.8520 & 0.8520 & 0.9515 & 0.7984 & 0.8520 & 0.8520 & 0.8520 & 1.0000\end{array}\right)$
$C^{8}$

Hence, $C^{8}$ is an equivalent correlation matrix.
Step 3. For a confidence level $\lambda_{0}$, to do clustering for HFSs, we construct a $\lambda_{0}$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} c_{i j}\right)_{7 \times 7}$, and based on which, we get the possible classifications of 10 firms $A_{i}(i=1,2, \cdots, 10)$, see Table 2.17 (Chen et al. 2013a).

Table 2.17. The clustering result of 10 firms

| Classes | Confidence levels | Hesitant fuzzy clustering algorithm |
| :---: | :---: | :---: |
| 10 | $0.9515<\lambda_{0} \leq 1$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\} \\ \left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 9 | $0.9306<\lambda_{0} \leq 0.9515$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\} \\ \left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 8 | $0.9238<\lambda_{0} \leq 0.9306$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{9}\right\},\left\{A_{6}\right\} \\ \left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 7 | $0.9104<\lambda_{0} \leq 0.9238$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{7}, A_{9}\right\} \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 6 | $0.9025<\lambda_{0} \leq 0.9104$ | $\begin{gathered} \left\{A_{1}, A_{6}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{7}, A_{9}\right\} \\ \left\{A_{8}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 5 | $0.8997<\lambda_{0} \leq 0.9025$ | $\begin{gathered} \left\{A_{1}, A_{6}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{7}, A_{8}, A_{9}\right\} \\ \left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 4 | $0.8520<\lambda_{0} \leq 0.8997$ | $\left\{A_{1}, A_{6}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{4}, A_{7}, A_{8}, A_{9}\right\},\left\{A_{5}, A_{10}\right\}$ |
| 3 | $0.8200<\lambda_{0} \leq 0.8520$ | $\left\{A_{1}, A_{6}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{4}, A_{5}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |
| 2 | $0.7984<\lambda_{0} \leq 0.8200$ | $\left\{A_{1}, A_{6}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |
| 1 | $0 \leq \lambda_{0} \leq 0.7984$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |

Under the group setting, the DMs' evaluation information usually does not reach an agreement for the objects that need to be classified. Examples 2.11 and 2.12 clearly show that the clustering algorithm based on HFSs provides a proper way to resolve this issue. However, it is interesting to point out that for these two real case studies, if adopting the conventional clustering methods within the framework of intuitionistic fuzzy sets and fuzzy sets, it needs to transform HFSs into fuzzy sets (or intuitionistic fuzzy sets), which gives rise to a difference in the accuracy of data in the two types, it will have an effect on the clustering results. We have actually performed such a clustering study by transforming the data in Tables 2.15 and 2.16 using intuitionistic fuzzy sets and fuzzy sets, respectively. We find that the results are different from those obtained by using HFSs, as expected.

### 2.5 Hesitant Fuzzy Agglomerative Hierarchical Clustering Algorithms

In the last decade, the research on clustering methods (Yang and Shih 2001; Fan et al. 2003; Mingoti and Lima 2006; Lu et al. 2008; Dong et al. 2006) has attracted considerable interest due to its increasing applications in various types of problems. The principle clustering approaches contain hierarchical algorithms, partitioning algorithms, and density-based algorithms, etc. The hierarchical clustering as a crucial method is either agglomerative or divisive. It consists of a sequence of iterative steps to partition at different layers. The layers are constructed by using merge-and-split techniques. Once a group of objects are merged or split, the next step will operate on the newly generated cluster (Lu et al. 2008). Hierarchical clustering algorithm gathers data to form a tree shaped structure, which is a widely used clustering technique and can be further divided into two categories: (1) Agglomerative methods, which proceed by making a series of merges of a collection of objects into more general groups; (2) Divisive methods, which separate the objects successively into finer groups (Tu et al. 2012). The agglomerative method has been more commonly used (Bordogna et al. 2006), which has been extensively discussed in the literature (Chaudhuri and Chaudhuri 1995; Yager 2000; Miyamoto 2003; Cui and Chae 2011; Cilibrasia and Vitányi 2011). However, most existing agglomerative hierarchical clustering algorithms are designed for clustering the real numbers and not suitable to cluster hesitant fuzzy information (Torra and Narukawa 2009; Torra 2010).

Based on the traditional agglomerative hierarchical clustering algorithm (Miyamoto 1990), Zhang and Xu (2013) developed a hesitant fuzzy agglomerative hierarchical (HFAH) clustering algorithm to do clustering under hesitant fuzzy environments, whose steps are as follows:
(Algorithm 2.4) (HFAH Clustering Algorithm)
Step 1. Let each of the objects $A_{i}(i=1,2, \cdots, n)$ be considered as a unique cluster $\left\{A_{1}\right\},\left\{A_{2}\right\}, \cdots,\left\{A_{n}\right\}$, and calculate the distance $d_{i j}=d\left(A_{i}, A_{j}\right)$ by Eq.(2.12) or (2.14) and get the hesitant fuzzy distance matrix $H=\left(h_{i j}\right)_{n \times n}$.

Step 2. In the hesitant fuzzy distance matrix $H=\left(h_{i j}\right)_{n \times n}$, we search the minimal distance $d\left(A_{i}, A_{j}\right)=\min _{\substack{1 \leq p, q \leq n \\ p \neq q}} d\left(A_{p}, A_{q}\right)$ and combine the clusters $A$ and $A_{i}$ to form a new cluster $A_{i j}$, and meanwhile calculate the center of $A_{i j}$ by using Eq.(1.33).

Step 3. Update the hesitant fuzzy distance matrix by computing the distances between the new cluster $A_{i j}$ and the other clusters.
Step 4. Repeat Steps 2 and 3 until all objects are in the one cluster.

For convenience, we express the clustering process of our algorithm above by the following flow chart (see Fig. 2.1) (Zhang and Xu 2013):


Fig.2.1. The process flow chart of the HFAH clustering algorithm
Zhang and Xu (2013) gave an example (adapted from Xu and Xia (2011b)) to illustrate and verify the HFAH Clustering clustering algorithm for HFSs:

Example 2.13 (Zhang and Xu 2013). Energy is an indispensable factor for the social and economic development of societies. Thus the correct energy policy affects economic development and environment, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) $A_{i}(i=1,2,3,4,5)$ to be invested, and four factors to be considered:
(1) $x_{1}$ : Technological; (2) $x_{2}$ : Environmental; (3) $x_{3}$ : Socio-political; (4) $x_{4}$ : Economic (more details about them can see Kahraman and Kaya 2010). The attribute weight vector is $w=(0.1,0.3,0.4,0.2)^{\mathrm{T}}$. Several DMs are invited to evaluate the performances of the five alternatives, and all possible evaluations for an alternative under each attribute can be considered as a HFE. The results evaluated by the DMs are listed as follows:
$A_{1}=\left\{\left\langle x_{1},\{0.4,0.5,0.7\}\right\rangle,\left\langle x_{2},\{0.5,0.8\}\right\rangle,\left\langle x_{3},\{0.6,0.7,0.9\}\right\rangle,\left\langle x_{4},\{0.5,0.6\}\right\rangle\right\}$
$A_{2}=\left\{\left\langle x_{1},\{0.6,0.7,0.8\}\right\rangle,\left\langle x_{2},\{0.5,0.6\}\right\rangle,\left\langle x_{3},\{0.4,0.6,0.7\}\right\rangle,\left\langle x_{4},\{0.4,0.5\}\right\rangle\right\}$
$\left.\left.A_{3}=\left\{<x_{1},\{0.6,0.8\}\right\rangle,\left\langle x_{2},\{0.2,0.3,0.5\}\right\rangle,\left\langle x_{3},\{0.4,0.6\}\right\rangle,<x_{4},\{0.5,0.7\}\right\rangle\right\}$
$A_{4}=\left\{\left\langle x_{1},\{0.5,0.6,0.7\}\right\rangle,\left\langle x_{2},\{0.4,0.5\}\right\rangle,\left\langle x_{3},\{0.8,0.9\}\right\rangle,\left\langle x_{4},\{0.3,0.4,0.5\}\right\rangle\right\}$
$A_{5}=\left\{\left\langle x_{1},\{0.6,0.7\}\right\rangle,\left\langle x_{2},\{0.5,0.7\}\right\rangle,\left\langle x_{3},\{0.7,0.8\}\right\rangle,\left\langle x_{4},\{0.2,0.3,0.4\}\right\rangle\right\}$

Obviously, the numbers of values in different HFEs are different. In order to more accurately calculate the distance between two HFEs, we should extend the shorter one until both of them have the same length when we compare them. Here, we consider that the DMs are pessimistic, and so we change the hesitant fuzzy data by adding the minimal values as below:
$A_{1}=\left\{\left\langle x_{1},\{0.4,0.5,0.7\}\right\rangle,\left\langle x_{2},\{0.5,0.5,0.8\}\right\rangle,\left\langle x_{3},\{0.6,0.7,0.9\}\right\rangle,\left\langle x_{4},\{0.5,0.5,0.6\}\right\rangle\right\}$
$\left.A_{2}=\left\{<x_{1},\{0.6,0.7,0.8\}\right\rangle,\left\langle x_{2},\{0.5,0.5,0.6\}\right\rangle,\left\langle x_{3},\{0.4,0.6,0.7\}\right\rangle,\left\langle x_{4},\{0.4,0.4,0.5\}\right\rangle\right\}$
$A_{3}=\left\{<x_{1},\{0.6,0.6,0.8\}>,<x_{2},\{0.2,0.3,0.5\}>,<x_{3},\{0.4,0.4,0.6\}>,<x_{4},\{0.5,0.5,0.7\}>\right\}$
$A_{4}=\left\{\left\langle x_{1},\{0.5,0.6,0.7\}\right\rangle,\left\langle x_{2},(0.4,0.4,0.5)\right\rangle,\left\langle x_{3},(0.8,0.8,0.9)\right\rangle,\left\langle x_{4},(0.3,0.4,0.5)\right\rangle\right\}$
$\left.A_{5}=\left\{<x_{1},\{0.6,0.6,0.7\}\right\rangle,\left\langle x_{2},\{0.5,0.5,0.7\}\right\rangle,\left\langle x_{3},\{0.7,0.7,0.8\}\right\rangle,\left\langle x_{4},\{0.2,0.3,0.4\}\right\rangle\right\}$

Then we proceed to utilize Algorithm 2.4 to group these energy projects $A_{i}(i=1,2, \cdots, 5)$ as follows:

Step 1. Let each of the energy projects $A_{i}(i=1,2, \cdots, 5)$ be considered as a unique cluster $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$, and calculate the distance $d_{i j}=d\left(A_{i}, A_{j}\right)$ by Eq.(2.14) and then get the hesitant fuzzy distance matrix $\Pi_{1}=\left(d_{i j}\right)_{5 \times 5}:$

$$
\Pi_{1}=\left(\begin{array}{lllll}
0.0000 & 0.1517 & 0.2324 & 0.1494 & 0.1438 \\
0.1517 & 0.0000 & 0.1643 & 0.2066 & 0.1703 \\
0.2324 & 0.1643 & 0.0000 & 0.2576 & 0.2394 \\
0.1494 & 0.2066 & 0.2576 & 0.0000 & 0.1278 \\
0.1438 & 0.1703 & 0.2394 & 0.1278 & 0.0000
\end{array}\right)
$$

Step 2. In the hesitant fuzzy distance matrix $\Pi_{1}$, we search the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d\left(A_{4}, A_{5}\right)=0.1278$, then combine $\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$ to form a new cluster $\left\{A_{4}, A_{5}\right\}$. So the energy projects $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following four clusters: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, and compute the center of each new cluster by using Eq.(1.33):

$$
\begin{aligned}
& \dot{c}\left\{A_{4}, A_{5}\right\}=A_{45}=f\left(A_{4}, A_{5}\right)=\left\{\left\langle x_{1},\{0.8,0.85,0.84,0.88,0.91\}\right\rangle,\right. \\
& \left.<x_{2},\{0.7,0.75,0.82,0.85\}\right\rangle,\left\langle x_{3},\{0.94,0.96,0.97,0.98\}\right\rangle, \\
& \left.\left.<x_{4},\{0.44,0.51,0.52,0.58,0.6,0.64,0.65,0.7\}\right\rangle\right\} \\
& \dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}\right\}=A_{2}, \dot{c}\left\{A_{3}\right\}=A_{3}
\end{aligned}
$$

Step 3. Update the hesitant fuzzy distance matrix by computing the distances between the cluster $\left\{A_{4}, A_{5}\right\}$ and the other clusters $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\}$, respectively.

Due to that the numbers of values in different HFEs are different, and according to the regulations given by Xu and Xia (2011b), we consider that the DMs are pessimistic, and change the hesitant fuzzy data $A_{1}, A_{2}$ and $A_{3}$ by adding the minimal values as below:

$$
\begin{aligned}
A_{1} & =\left\{<x_{1},\{0.4,0.4,0.4,0.5,0.7\}>,<x_{2},\{0.5,0.5,0.5,0.8\}>\right. \\
& \left.<x_{3},\{0.6,0.6,0.7,0.9\}>,<x_{4},\{0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.6\}>\right\} \\
A_{2} & =\left\{<x_{1},\{0.6,0.6,0.6,0.7,0.8\}>,<x_{2},\{0.5,0.5,0.5,0.6\}>\right. \\
& \left.<x_{3},\{0.4,0.4,0.6,0.7\}>,<x_{4},\{0.4,0.4,0.4,0.4,0.4,0.4,0.4,0.5\}>\right\} \\
A_{3} & =\left\{<x_{1},\{0.6,0.6,0.6,0.6,0.8\}>,<x_{2},\{0.2,0.2,0.3,0.5\}>\right. \\
& \left.<x_{3},\{0.4,0.4,0.4,0.6\}>,<x_{4},\{0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.7\}>\right\}
\end{aligned}
$$

Then we compute the distances between $A_{45}$ and $A_{1}, A_{2}, A_{3}$, respectively, and get $d\left(A_{1}, A_{45}\right)=0.2071, d\left(A_{2}, A_{45}\right)=0.2649, d\left(A_{3}, A_{45}\right)=0.3596$, and update the hesitant fuzzy distance matrix as follow:

$$
\Pi_{2}=\left(\begin{array}{llll}
0.0000 & 0.1517 & 0.2324 & 0.2071 \\
0.1517 & 0.0000 & 0.1643 & 0.2649 \\
0.2324 & 0.1643 & 0.0000 & 0.3596 \\
0.2071 & 0.2649 & 0.3596 & 0.0000
\end{array}\right)
$$

Step 4. Check whether all objects are in the one cluster; If not, then we repeat Steps 2 and 3.

Since there are still four clusters $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, we repeat Steps 2 and 3 as follows:

In the hesitant fuzzy distance matrix $\Pi_{2}$, we find the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d\left(A_{1}, A_{2}\right)=0.1527$, then combine $\left\{A_{1}\right\}$ and $\left\{A_{2}\right\}$ to form a new cluster $\left\{A_{1}, A_{2}\right\}$. So the energy projects $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following three clusters $\left\{A_{1}, A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, and then compute the center of each new cluster by using Eq.(1.33):

$$
\begin{aligned}
& \dot{c}\left\{A_{1}, A_{2}\right\}=A_{12}=f\left(A_{1}, A_{2}\right) \\
& =\left\{<x_{1},\{0.76,0.8,0.82,0.88,0.85,0.9,0.91,0.94\}>\right. \\
& \quad<x_{2},\{0.75,0.8,0.9,0.92\}> \\
& \quad<x_{3},\{0.76,0.82,0.84,0.88,0.91,0.94,0.96,0.97\}> \\
& \left.\quad<x_{4},\{0.7,0.75,0.75,0.76,0.8\}>\right\} \\
& \dot{c}\left\{A_{4}, A_{5}\right\}=A_{45}, \dot{c}\left\{A_{3}\right\}=A_{3}
\end{aligned}
$$

After that, we continue to compute the distances between $\left\{A_{4}, A_{5}\right\}$ and $\left\{A_{1}, A_{2}\right\},\left\{A_{1}, A_{2}\right\}$ and $\left\{A_{3}\right\}$, as well as $\left\{A_{4}, A_{5}\right\}$ and $\left\{A_{3}\right\}$, then we get

$$
\begin{gathered}
d\left(\dot{c}\left\{A_{1}, A_{2}\right\}, \dot{c}\left\{A_{3}\right\}\right)=0.434, d\left(\dot{c}\left\{A_{1}, A_{2}\right\}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.0967 \\
d\left(\dot{c}\left\{A_{3}\right\}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3596
\end{gathered}
$$

and update the hesitant fuzzy distance matrix as follow:

$$
\Pi_{3}=\left(\begin{array}{lll}
0.0000 & 0.4340 & 0.0967 \\
0.4340 & 0.0000 & 0.3596 \\
0.0967 & 0.3596 & 0.0000
\end{array}\right)
$$

However, there are still three clusters $\left\{A_{1}, A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, so we continue to repeat Steps 2 and 3 as follows:

In the hesitant fuzzy distance matrix $\Pi_{3}$, we can find the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d\left(A_{12}, A_{45}\right)=0.0967$, then combine $\left\{A_{1}, A_{2}\right\}$ and $\left\{A_{4}, A_{5}\right\}$ to form a new cluster $A_{1245}=\left\{A_{1}, A_{2}, A_{4}, A_{5}\right\}$. So the energy projects $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following two clusters $\left\{A_{3}\right\},\left\{A_{1}, A_{2}, A_{4}, A_{5}\right\}$.


Fig.2.2. Classification of the energy projects $A_{i}(i=1,2,3,4,5)$

Finally, the above two clusters can be further clustered into a unique cluster $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$, and all the above processes can be shown as in Fig. 2.2 (Zhang and Xu 2013).

In order to compare with the intuitionistic fuzzy hierarchical (IFH) clustering algorithm (Xu 2009a), we consider the HFSs' envelopes, i.e., intuitionistic fuzzy data (Chen 2012), and make intuitionistic fuzzy hierarchical clustering analysis:

According to Definition 1.6, the IFN $\alpha_{e n v}(h)$ is the envelope of the HFE $h$, then we can transform the hesitant fuzzy data of Example 2.13 into the intuitionistic fuzzy data $A_{i}=\left\{<\mu\left(x_{j}\right), v\left(x_{j}\right)>\mid x_{j} \in X\right\}(i=1,2,3,4,5)$, where
$\left.\left.\left.A_{1}=\left\{<x_{1},(0.4,0.3)\right\rangle,\left\langle x_{2},(0.5,0.2)\right\rangle,<x_{3},(0.6,0.1)\right\rangle,<x_{4},(0.5,0.4)\right\rangle\right\}$
$\left.A_{2}=\left\{\left\langle x_{1},(0.6,0.2)\right\rangle,\left\langle x_{2},(0.5,0.4)\right\rangle,<x_{3},(0.4,0.3)\right\rangle,\left\langle x_{4},(0.4,0.5)\right\rangle\right\}$
$A_{3}=\left\{\left\langle x_{1},(0.6,0.2)\right\rangle,\left\langle x_{2},(0.2,0.5)\right\rangle,\left\langle x_{3},(0.4,0.6)\right\rangle,\left\langle x_{4},(0.5,0.3)\right\rangle\right\}$
$A_{4}=\left\{\left\langle x_{1},(0.5,0.3)\right\rangle,\left\langle x_{2},(0.4,0.5)\right\rangle,\left\langle x_{3},(0.8,0.1)\right\rangle,\left\langle x_{4},(0.3,0.5)\right\rangle\right\}$
$A_{5}=\left\{\left\langle x_{1},(0.6,0.3)\right\rangle,\left\langle x_{2},(0.5,0.3)\right\rangle,\left\langle x_{3},(0.7,0.2)\right\rangle,\left\langle x_{4},(0.2,0.6)\right\rangle\right\}$
and then the energy projects $A_{i}(i=1,2,3,4,5)$ can be clustered as the following intuitionistic fuzzy hierarchical clustering algorithm (Xu 2009a):

Step 1. Let each of the energy projects $A_{i}(i=1,2,3,4,5)$ be considered as a unique cluster $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$, and calculate the distance $d_{i j}=d\left(A_{i}, A_{j}\right)$ by the following distance measure:
$d\left(A_{i}, A_{j}\right)=\sqrt{\frac{1}{2} \sum_{k=1}^{n} w_{k}\left(\left(\mu_{A_{i}}\left(x_{k}\right)-\mu_{A_{j}}\left(x_{k}\right)\right)^{2}+\left(v_{A_{i}}\left(x_{k}\right)-v_{A_{j}}\left(x_{k}\right)\right)^{2}+\left(\pi_{A_{i}}\left(x_{k}\right)-\pi_{A_{j}}\left(x_{k}\right)\right)^{2}\right)}$
where the weight vector of the attributes $x_{j}(j=1,2,3,4)$ is $w=(0.1,0.3,0.4,0.2)^{\mathrm{T}}$, and we get the intuitionistic fuzzy distance matrix $\Pi_{1}=\left(d_{i j}\right)_{5 \times 5}:$

$$
\Pi_{1}=\left(\begin{array}{lllll}
0.0000 & 0.1817 & 0.2449 & 0.2098 & 0.2214 \\
0.1817 & 0.0000 & 0.1761 & 0.2324 & 0.2145 \\
0.2449 & 0.1761 & 0.0000 & 0.2702 & 0.2608 \\
0.2098 & 0.2324 & 0.2702 & 0.0000 & 0.1549 \\
0.2214 & 0.2145 & 0.2608 & 0.1549 & 0.0000
\end{array}\right)
$$

Step 2. In the intuitionistic fuzzy distance matrix $\Pi_{1}$, we search the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d_{5}\left(A_{4}, A_{5}\right)=0.1278$, and then combine $\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$ to form a new cluster $\left\{A_{4}, A_{5}\right\}$; So the energy projects $A_{j}(j=1,2,3,4,5)$ can be clustered into the following four clusters: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, and compute the center of each new cluster by the following average operation :

$$
\begin{equation*}
f\left(A_{1}, A_{2}, \cdots, A_{n}\right)=\left\{<x_{j}, 1-\prod_{i=1}^{n}\left(1-\mu_{A_{i}}\left(x_{j}\right)\right)^{\frac{1}{n}}, \prod_{i=1}^{n}\left(v_{A_{i}}\left(x_{j}\right)\right)^{\frac{1}{n}}>\mid x_{j} \in X\right\} \tag{2.187}
\end{equation*}
$$

So we can get the center of each new cluster:

$$
\begin{aligned}
& \begin{aligned}
& \dot{c}\left\{A_{4}, A_{5}\right\}=A_{45}=f\left(A_{4}, A_{5}\right) \\
&=\left\{<x_{1},(0.5528,0.3)>,<x_{2},(0.4523,0.3873)>\right. \\
&\left.<x_{3},(0.7551,0.1414)>,<x_{4},(0.2517,0.5477)>\right\} \\
& \dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}\right\}=A_{2}, \dot{c}\left\{A_{3}\right\}=A_{3}
\end{aligned}
\end{aligned}
$$

Step 3. Update the intuitionistic fuzzy distance matrix by computing the distances between the cluster $\left\{A_{4}, A_{5}\right\}$ and the other clusters $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\}$, respectively.

$$
\Pi_{2}=\left(\begin{array}{llll}
0.0000 & 0.1817 & 0.2449 & 0.1819 \\
0.1817 & 0.0000 & 0.1761 & 0.2075 \\
0.2449 & 0.1761 & 0.0000 & 0.2605 \\
0.1819 & 0.2075 & 0.2605 & 0.0000
\end{array}\right)
$$

Step 4. Repeat Steps 2 and 3 until only one cluster remains.
Obviously, there are still four clusters $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, we repeat Steps 2 and 3:

In the intuitionistic fuzzy distance matrix $\Pi_{2}$, we find the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d_{5}\left(A_{2}, A_{3}\right)=0.1761$, then combine $\left\{A_{2}\right\}$ and $\left\{A_{3}\right\}$ to form a new cluster $\left\{A_{2}, A_{3}\right\}$. So the energy projects $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following three clusters $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, and then compute the center of $\left\{A_{2}, A_{3}\right\}$ by using Eq.(2.187):

$$
\begin{aligned}
& \dot{c}\left\{A_{2}, A_{3}\right\}=A_{23}=f\left(A_{2}, A_{3}\right) \\
& \quad=\left\{<x_{1},(0.6,0.2)>,<x_{2},(0.3675,0.4472)>,<x_{3},(0.4,0.3464)>\right. \\
& \left.\quad<x_{4},(0.4523,0.3873)>\right\} \\
& \dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{4}, A_{5}\right\}=A_{45}
\end{aligned}
$$

After that, we compute the distances between $\left\{A_{4}, A_{5}\right\}$ and $\left\{A_{2}, A_{3}\right\}$, $\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{1}\right\}$, as well as $\left\{A_{4}, A_{5}\right\}$ and $\left\{A_{1}\right\}$, and update the intuitionistic fuzzy distance matrix as follow:

$$
\Pi_{3}=\left(\begin{array}{lll}
0.0000 & 0.1948 & 0.1819 \\
0.1948 & 0.0000 & 0.2176 \\
0.1819 & 0.2176 & 0.0000
\end{array}\right)
$$

However, there are still three clusters $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}, A_{5}\right\}$, so we continue to repeat Steps 2 and 3 as follows:

In the intuitionistic fuzzy distance matrix $\Pi_{3}$, we find the smallest distance $d_{\text {min }}\left(A_{i}, A_{j}\right)=d_{5}\left(A_{1}, A_{45}\right)=0.1819$, then combine $\left\{A_{1}\right\}$ and $\left\{A_{4}, A_{5}\right\}$ to form a new cluster $A_{145}=\left\{A_{1}, A_{4}, A_{5}\right\}$. So the energy projects $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following two clusters $\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{1}, A_{4}, A_{5}\right\}$.


Fig.2.3. Classification of the energy projects $A_{i}(i=1,2,3,4,5)$

At length, the above two clusters $\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{1}, A_{4}, A_{5}\right\}$ can be further clustered into a unique cluster $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$, and all the above processes can be shown as in Fig. 2.3 (Zhang and Xu 2013).

Moreover, Miyamoto (2003) also proposed a fuzzy multiset model for information clustering and applied this model in information retrieval on the World Wide Web. In his research, three classical clustering methods including the hard c-means algorithm, the fuzzy c-means algorithm and the agglomerative hierarchical algorithm were extended into clustering the fuzzy multiset information. Here we just make a comparison with the fuzzy multiset agglomerative hierarchical (FMAH) algorithm of Miyamoto (2003), which is the closest to Algorithm 2.4. Namely, we consider that the data of Example 2.13 are expressed by fuzzy multisets instead of HFSs. As Torra (2010) pointed out that all HFSs can be represented as fuzzy multisets, thus we also regard the data information of Example 2.13 as fuzzy multiset information and utilize the FMAH clustering algorithm (Miyamoto 2003) to group these energy projects $A_{i}(i=1,2, \cdots, 5)$, then we have the following clustering results as in Table 2.18 (In order to provide a better view of the comparison results, we also put the clustering results of the other two algorithms into Table 2.18).

Table 2.18. Clustering results

| Classes | HFAH clustering Algorithm | IFH clustering algorithm | FMAH clustering algorithm |
| :---: | :---: | :---: | :--- |
| 5 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ |
| 4 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ |
| 3 | $\left\{A_{1}, A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{3}\right\},\left\{A_{2}, A_{4}, A_{5}\right\}$ |
| 2 | $\left\{A_{3}\right\},\left\{A_{1}, A_{2}, A_{4}, A_{5}\right\}$ | $\left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{4}, A_{5}\right\}$ | $\left\{A_{3}\right\},\left\{A_{1}, A_{2}, A_{4}, A_{5}\right\}$ |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ |

In Table 2.18, we see that the clustering results between the HFAH clustering algorithm and the IFH algorithm are quite different. The main reason is that the HFAH clustering algorithm clusters the fuzzy information which is represented by several possible values, not by a margin of error (as in intuitionistic fuzzy sets), while if adopting the intuitionistic fuzzy hierarchical clustering algorithm, it needs to transform HFSs into intuitionistic fuzzy sets, which gives rise to a difference in the accuracy of data in the two types, it will have an effect on the clustering results. Meanwhile, through Table 2.18, we also note that the clustering results of the FMAH algorithm are different from the results derived by the HFAH algorithm. The reason is that although HFS can be represented as fuzzy multisets, their interpretations (practical significances) and their operations are different (Torra 2010). Thus, we cannot apply directly the IFH (or FMAH) algorithm to cluster data represented by HFSs. Apparently, when we meet some situations where the information is represented by several possible values, the HFAH clustering algorithm demonstrates its great superiority in clustering those hesitant fuzzy data.

Nevertheless, we cannot claim that our method produces a better solution for different clustering data because different clustering methods have their own advantages in dealing with different types of data. For the sake of knowing the
advantages of Algorithm 2.4 over the existing algorithms which also handle hesitant fuzzy information, we furthermore compare our technique with Chen et al. (2013a)'s method using another example (adapted from Chen et al. (2013a)):

Example 2.14 (Zhang and Xu 2013). The assessment of business failure risk, i.e., the assessment of enterprise performance and the prediction of failure events has drawn the attention of many researchers in recent years. For this purpose, ten enterprises $A_{i}(i=1,2, \ldots, 10)$ evaluated on five attributes ( $x_{1}$ : managers work experience, $x_{2}$ : profitability, $x_{3}$ : operating capacity, $x_{4}$ : debt-paying ability, and $x_{5}$ : market competition) will be classified according to their risk of failure and also assume that the weighting vector of five attributes is $w=(0.15,0.3,0.2,0.25,0.1)^{\mathrm{T}}$. In order to better make the assessment, several risk evaluation organizations are requested. The normalized evaluation data, represented by HFEs, are displayed in Table 2.19 (Zhang and Xu 2013):

Table 2.19. The evaluation information for the five attributes of ten enterprises

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.3,0.4,0.5\}$ | $\{0.4,0.5\}$ | $\{0.8\}$ | $\{0.5\}$ | $\{0.2,0.3\}$ |
| $A_{2}$ | $\{0.4,0.6\}$ | $\{0.6,0.8\}$ | $\{0.2,0.3\}$ | $\{0.3,0.4\}$ | $\{0.6,0.7,0.9\}$ |
| $A_{3}$ | $\{0.5,0.7\}$ | $\{0.9\}$ | $\{0.3,0.4\}$ | $\{0.3\}$ | $\{0.8,0.9\}$ |
| $A_{4}$ | $\{0.3,0.4,0.5\}$ | $\{0.8,0.9\}$ | $\{0.7,0.9\}$ | $\{0.1,0.2\}$ | $\{0.9,1.0\}$ |
| $A_{5}$ | $\{0.8,1.0\}$ | $\{0.8,1.0\}$ | $\{0.4,0.6\}$ | $\{0.8\}$ | $\{0.7,0.8\}$ |
| $A_{6}$ | $\{0.4,0.5,0.6\}$ | $\{0.2,0.3\}$ | $\{0.9,1.0\}$ | $\{0.5\}$ | $\{0.3,0.4,0.5\}$ |
| $A_{7}$ | $\{0.6\}$ | $\{0.7,0.9\}$ | $\{0.8\}$ | $\{0.3,0.4\}$ | $\{0.4,0.7\}$ |
| $A_{8}$ | $\{0.9,1.0\}$ | $\{0.7,0.8\}$ | $\{0.4,0.5\}$ | $\{0.5,0.6\}$ | $\{0.7\}$ |
| $A_{9}$ | $\{0.4,0.6\}$ | $\{1.0\}$ | $\{0.6,0.7\}$ | $\{0.2,0.3\}$ | $\{0.9,1.0\}$ |
| $A_{10}$ | $\{0.9\}$ | $\{0.6,0.7\}$ | $\{0.5,0.8\}$ | $\{1.0\}$ | $\{0.7,0.8,0.9\}$ |

With the HFAH clustering algorithm, we have the following clustering results as in Table 2.20. However, if we use Chen et al. (2013a)'s method (Algorithm-HFSC), we first need to construct the hesitant fuzzy correlation matrix based on the data in Table 2.19:

$$
C=\left(\begin{array}{llllllllll}
1.0000 & 0.7984 & 0.6583 & 0.6635 & 0.5964 & 0.9104 & 0.7572 & 0.6761 & 0.6147 & 0.5983 \\
0.7984 & 1.0000 & 0.8200 & 0.7139 & 0.6459 & 0.6666 & 0.7411 & 0.7458 & 0.7052 & 0.5855 \\
0.6583 & 0.8200 & 1.0000 & 0.8813 & 0.7593 & 0.6082 & 0.8997 & 0.8872 & 0.8683 & 0.6757 \\
0.6635 & 0.7139 & 0.8813 & 1.0000 & 0.7423 & 0.6542 & 0.9238 & 0.8743 & 0.9306 & 0.6742 \\
0.5964 & 0.6459 & 0.7593 & 0.7423 & 1.0000 & 0.5761 & 0.7737 & 0.8520 & 0.8253 & 0.9515 \\
0.9104 & 0.6666 & 0.6082 & 0.6542 & 0.5761 & 1.0000 & 0.7427 & 0.6647 & 0.5816 & 0.6124 \\
0.7572 & 0.7411 & 0.8997 & 0.9238 & 0.7737 & 0.7427 & 1.0000 & 0.9025 & 0.8723 & 0.7217 \\
0.6761 & 0.7458 & 0.8872 & 0.8743 & 0.8520 & 0.6647 & 0.9025 & 1.0000 & 0.8617 & 0.8067 \\
0.6147 & 0.7052 & 0.8683 & 0.9306 & 0.8253 & 0.5816 & 0.8723 & 0.8617 & 1.0000 & 0.7377 \\
0.5983 & 0.5855 & 0.6757 & 0.6742 & 0.9515 & 0.6124 & 0.7217 & 0.8067 & 0.7377 & 1.0000
\end{array}\right)
$$

In order to get the clustering result with Chen et al. (2013a)'s method, we should get the equivalent correlation matrix. By the composition operation of correlation matrices, we have

$$
C^{16}=C^{8} \circ C^{8}
$$

$=\left(\begin{array}{llllllllll}1.0000 & 0.7984 & 0.7984 & 0.7984 & 0.7984 & 0.9104 & 0.7984 & 0.7984 & 0.7984 & 0.7984 \\ 0.7984 & 1.0000 & 0.8200 & 0.8200 & 0.8200 & 0.7984 & 0.8200 & 0.8200 & 0.8200 & 0.8200 \\ 0.7984 & 0.8200 & 1.0000 & 0.8997 & 0.8520 & 0.7984 & 0.8997 & 0.8997 & 0.8997 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 1.0000 & 0.8520 & 0.7984 & 0.9238 & 0.9025 & 0.9306 & 0.8520 \\ 0.7984 & 0.8200 & 0.8520 & 0.8520 & 1.0000 & 0.7984 & 0.8520 & 0.8520 & 0.8520 & 0.9515 \\ 0.9104 & 0.7984 & 0.7984 & 0.7984 & 0.7984 & 1.0000 & 0.7984 & 0.7984 & 0.7984 & 0.7984 \\ 0.7984 & 0.8200 & 0.8997 & 0.9238 & 0.8520 & 0.7984 & 1.0000 & 0.9025 & 0.9238 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 0.9025 & 0.8520 & 0.7984 & 0.9025 & 1.0000 & 0.9025 & 0.8520 \\ 0.7984 & 0.8200 & 0.8997 & 0.9306 & 0.8520 & 0.7984 & 0.9238 & 0.9025 & 1.0000 & 0.8520 \\ 0.7984 & 0.8200 & 0.8520 & 0.8520 & 0.9515 & 0.7984 & 0.8520 & 0.8520 & 0.8520 & 1.0000\end{array}\right)$
$=C^{8}$

Then, we can make clustering analysis with Chen et al. (2013a)'s method and at the same time get the possible classifications of ten firms $A_{i}(i=1,2, \cdots, 10)$, listed in Table 2.20 (Zhang and Xu 2013).

Table 2.20. Clustering results

| Classes | HFAH clustering Algorithm | Algorithm-HFSC |
| :---: | :---: | :---: |
| 10 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\}, \\ \left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\}, \\ \left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 9 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\}, \\ \left\{A_{8}\right\},\left\{A_{10}\right\},\left\{A_{4}, A_{9}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\}, \\ \left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 8 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\} \\ \left\{A_{4}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\}, \\ \left\{A_{4}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 7 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\}, \\ \left\{A_{4}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{6}\right\},\left\{A_{8}\right\}, \\ \left\{A_{4}, A_{7}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 6 | $\begin{gathered} \left\{A_{2}, A_{3}\right\},\left\{A_{1}\right\},\left\{A_{6}\right\},\left\{A_{8}\right\}, \\ \left\{A_{4}, A_{7}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{8}\right\},\left\{A_{1}, A_{6}\right\} \\ \left\{A_{4}, A_{7}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 5 | $\begin{gathered} \left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{6}\right\},\left\{A_{8}\right\}, \\ \left\{A_{4}, A_{7}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{1}, A_{6}\right\}, \\ \left\{A_{4}, A_{7}, A_{8}, A_{9}\right\},\left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 4 | $\left\{A_{1}, A_{6}\right\},\left\{A_{8}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{7}, A_{9}\right\},\left\{A_{5}, A_{10}\right\}$ | $\begin{gathered} \left\{A_{2}\right\},\left\{A_{1}, A_{6}\right\},\left\{A_{3}, A_{4}, A_{7}, A_{8}, A_{9}\right\}, \\ \left\{A_{5}, A_{10}\right\} \end{gathered}$ |
| 3 | $\left\{A_{1}, A_{6}\right\},\left\{A_{8}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{7}, A_{9}, A_{10}\right\}$ | $\begin{gathered} \left\{A_{2}\right\},\left\{A_{1}, A_{6}\right\}, \\ \left\{A_{3}, A_{4}, A_{5}, A_{7}, A_{8}, A_{9}, A_{10}\right\} \end{gathered}$ |
| 2 | $\left\{A_{1}, A_{6}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ | $\left\{A_{1}, A_{6}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |

We can see from Table 2.20 that the clustering results of our algorithm are very little different from the results by using the Algorithm-HFSC (Chen et al. 2013a). Especially when the numbers of clusters of the clustering results in Example 2.14 are equal to $k(k=1,2,8,10)$, the results of those two algorithms are exactly the same. Meanwhile, it is not hard to see that our algorithm has some desirable advantages over Algorithm-HFSC. Firstly, the HFAH clustering algorithm requires much less computational efforts. Because it is based on the conventional agglomerative hierarchical clustering procedure, the hesitant fuzzy aggregation operator, and the hesitant fuzzy distance measure, which does not need to calculate the equivalent association matrix, while Chen et al. (2013a)'s method, as its calculation process mentioned above, needs to transform the hesitant fuzzy correlation matrix into the equivalent correlation matrix, which requires lots of computational efforts. Let $m$ and $n$ represent the number of alternatives and attributes, respectively. Then the computational complexities of our method and Chen et al. (2013a)'s method are $O\left(m n+12 n^{2}\right)$ and $O\left(m n+12 n^{2}+k n^{2}\right)$, respectively, where $k(k \geq 2)$ represents the transfer times until we get the equivalent matrix. The elapsed time may become closed as $n$ increases. Considering the practical application, we think the HFAH clustering algorithm can save much more time and computational efforts. Secondly, the Algorithm-HFSC (Chen et al. 2013a) needs to transform the hesitant fuzzy association coefficients matrix into a hesitant fuzzy equivalent association matrix, some information maybe missing during this process, namely, the hesitant fuzzy equivalent association matrix cannot reflect all the information that the hesitant fuzzy association coefficients matrix contains. Thus, we can very confidently say that the HFAH clustering algorithm makes the clustering process more effective and needs less computational efforts, and meanwhile, the HFAH clustering algorithm offers a flexible, non-parametric approach for clustering HFSs. In addition, the ultimate clustering results of Algorithm 2.4 can be represented by the dendrogram, which provides very informative description and a visualization of the potential data clustering structures, especially when real hierarchical relations exist in the data, such as the data from evolutionary research on different species of organisms, or other applications in medicine, biology and archaeology (Everitt et al. 2001).

### 2.6 Hierarchical Hesitant Fuzzy K-means Clustering Algorithm

K-means is one of the latter principle representatives. The procedure of K-means algorithm is as follows (Pena et al. 1999): First, select somehow an initial partition of the database in the K clusters and calculate the centroid of each cluster, or select the initial seeds randomly, and then all the objects are compared with each centroid by means of the distance and assigned to the closest cluster. The above
steps are repeated many times until the changes in the cluster centers from one stage to the next are close to zero or smaller than a pre-specified value. K-means algorithm (HilalInana and Kuntalp 2007; Pena et al. 1999; Tokushige et al. 2007; Pop and Sarbu 1997) is of robustness, and its sensitivity to initial environment, i.e., initial cluster or initial seeds, has been discussed (Sun et al. 2002; Mingoti and Lima 2006; Pena et al. 1999; Khan and Ahmad 2004). At present, clustering algorithms usually assume that appropriate initial cluster centers can be found in advance, however, there has been, as of now, no universal methods to determine the initial cluster centers (Pena et al. 1999). In this section we shall utilize the results of hierarchical clustering as an initial cluster for HFSs.
(Algorithm 2.5) (K-Means Clustering) (Chen et al. 2014)

Step 1. Give the number of cluster.

Step 2. Select the results of Algorithm 2.4 as initial clusters, and calculate initial cluster centroids by Eq.(1.33).

Step 3. Calculate the distances between HFSs $A_{i}(i=1,2, \cdots, n)$ and centroids by

Eq.(2.12) or Eq.(2.14); Assign $A_{i}$ to the closest centroid.

Step 4. Recalculate the centroids of the clusters.

Step 5. Repeat Steps 2 and 3 until the centroids stabilize.

Example 2.15 (Chen et al. 2014). An enterprise puts forward five kinds of marketing programs $A_{i}(i=1,2, \cdots, 5)$ for new products. Some DMs evaluate these programs from eight aspects on the basis of their familiar fields. These eight aspects are denoted with the feature space $X=\left\{x_{1}, x_{2}, \cdots, x_{8}\right\}$, and their weight vector is $w=(0.15,0.10,0.12,0.15,0.10,0.13,0.14,0.11)^{\mathrm{T}}$. Suppose that various DMs give different values for a certain attribute of a program, we employ HFSs (for convenience, we also denote them by $A_{i}(i=1,2, \cdots, 5)$ ) to represent the evaluated information over the five kinds of marketing programs. The corresponding data are listed in Table 2.21 (Chen et al. 2014).

Table 2.21. Hesitant fuzzy evaluated information

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.3 .0 .5\}$ | $\{0.1,0.2\}$ | $\{0.5,0.6,0.7\}$ | $\{0.9,0.95,1\}$ |
| $A_{2}$ | $\{0.5,0.6\}$ | $\{0.6,0.7,0.85\}$ | $\{1\}$ | $\{0.15,0.2,0.35\}$ |
| $A_{3}$ | $\{0.45,0.5,0.65\}$ | $\{0.6,0.7\}$ | $\{0.9,0.95,1\}$ | $\{0.1,0.15,0.2\}$ |
| $A_{4}$ | $\{1\}$ | $\{1\}$ | $\{0.85,0.9\}$ | $\{0.75,0.8,0.85\}$ |
| $A_{5}$ | $\{0.9,0.95,1\}$ | $\{0.9\}$ | $\{0.8,0.85,0.9\}$ | $\{0.7,0.75,0.8\}$ |
|  |  |  |  |  |
|  | $x_{5}$ | $x_{6}$ | $x_{7}$ | $x_{8}$ |
| $A_{1}$ | $\{0.4,0.5,0.65\}$ | $\{0.1\}$ | $\{0.3,0.4,0.5\}$ | $\{1\}$ |
| $A_{2}$ | $\{0,0.1,0.2\}$ | $\{0.7,0.8,0.85\}$ | $\{0.5,0.6,0.7\}$ | $\{0.65,0.7,0.8\}$ |
| $A_{3}$ | $\{0.2,0.3\}$ | $\{0.6,0.7,0.8\}$ | $\{0.15,0.2\}$ | $\{0.2,0.3,0.35\}$ |
| $A_{4}$ | $\{0.2\}$ | $\{0.5,0.6,0.85\}$ | $\{0.3,0.35\}$ | $\{0.15,0.2,0.25\}$ |

Then we use first use the agglomerative hierarchical clustering to classify the five types of marketing programs:

Step 1. In this step, each of the HFSs $A_{j}(j=1,2, \cdots, 5)$ is considered as a unique cluster: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$.

Step 2. Compare each of the HFSs $A_{j}(j=1,2, \cdots, 5)$ with all the other four HFSs using the hesitant weighted Hamming distance (2.12):

$$
\begin{aligned}
& d_{12}\left(A_{1}, A_{2}\right)=d_{12}\left(A_{2}, A_{1}\right)=0.4335, d_{12}\left(A_{1}, A_{3}\right)=d_{12}\left(A_{3}, A_{1}\right)=0.4598 \\
& d_{12}\left(A_{1}, A_{4}\right)=d_{12}\left(A_{4}, A_{1}\right)=0.3827, d_{12}\left(A_{1}, A_{5}\right)=d_{12}\left(A_{5}, A_{1}\right)=0.3494 \\
& d_{12}\left(A_{2}, A_{3}\right)=d_{12}\left(A_{3}, A_{2}\right)=0.1643, d_{12}\left(A_{2}, A_{4}\right)=d_{12}\left(A_{4}, A_{2}\right)=0.3900 \\
& d_{12}\left(A_{2}, A_{5}\right)=d_{12}\left(A_{5}, A_{2}\right)=0.3682, d_{12}\left(A_{3}, A_{4}\right)=d_{12}\left(A_{4}, A_{3}\right)=0.3038 \\
& d_{12}\left(A_{3}, A_{5}\right)=d_{12}\left(A_{5}, A_{3}\right)=0.3132, d_{12}\left(A_{4}, A_{5}\right)=d_{12}\left(A_{5}, A_{4}\right)=0.1251
\end{aligned}
$$

Then
$d_{12}\left(A_{1}, A_{5}\right)=\min \left\{d_{12}\left(A_{1}, A_{2}\right), d_{12}\left(A_{1}, A_{3}\right), d_{12}\left(A_{1}, A_{4}\right), d_{12}\left(A_{1}, A_{5}\right)\right\}=0.3494$
$d_{12}\left(A_{2}, A_{3}\right)=\min \left\{d_{12}\left(A_{2}, A_{1}\right), d_{12}\left(A_{2}, A_{3}\right), d_{12}\left(A_{2}, A_{4}\right), d_{12}\left(A_{2}, A_{5}\right)\right\}=0.1643$
$d_{12}\left(A_{4}, A_{5}\right)=\min \left\{d_{12}\left(A_{4}, A_{1}\right), d_{12}\left(A_{4}, A_{2}\right), d_{12}\left(A_{4}, A_{3}\right), d_{12}\left(A_{4}, A_{5}\right)\right\}=0.1251$

Considering that only two clusters can be jointed in each step, the HFSs $A_{j}(j=1,2, \cdots, 5)$ are thus clustered into the following four clusters: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$.

Step 3. Calculate the center of each cluster using Eq.(1.33):

$$
\dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}\right\}=A_{2}, \dot{c}\left\{A_{3}\right\}=A_{3}
$$

$$
\dot{c}\left\{A_{4}, A_{5}\right\}=f\left(A_{4}, A_{5}\right\}
$$

$$
=\left\{\left\langle x_{1},\{1\}\right\rangle,\left\langle x_{2},\{1\}\right\rangle,\left\langle x_{3},\{0.9,0.8586,0.8775,0.85,0.8268\}\right\rangle,\right.
$$

$<x_{4},\{0.8268,0.8064,0.7879,0.8,0.7551,0.7764,0.75,0.7261\}>$, $\left\langle x_{5},\{0.6536,0.4343,0.3675\}\right\rangle,\left\langle x_{6},\{0.2567,0.2286\}\right\rangle$,
$<x_{7},\{0.2754,0.2517,0.2286,0.2254,0.2,0.1754,0.1784,0.1515,0.1254\}>$, $\left.<x_{8},\{0.5417,0.4084,0.3519\}>\right\}$

Comparing each cluster with the other three clusters with the hesitant weighted Hamming distance (Eq.(2.12)), we have

$$
\begin{gathered}
d_{12}\left(A_{1}, A_{2}\right)=d_{12}\left(A_{2}, A_{1}\right)=0.4335, d_{12}\left(A_{1}, A_{3}\right)=d_{12}\left(A_{3}, A_{1}\right)=0.4598 \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{4}, A_{5}\right\}, A_{1}\right)=0.3450, \\
d_{12}\left(A_{2}, A_{3}\right)=d_{12}\left(A_{3}, A_{2}\right)=0.1643 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{4}, A_{5}\right\}, A_{2}\right)=0.3889 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{4}, A_{5}\right\}, A_{3}\right)=0.3211
\end{gathered}
$$

Then the HFSs $A_{j}(j=1,2, \cdots, 5)$ are clustered into the following three clusters: $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$.

Step 4. Calculate the center of each cluster by using Eq.(1.33):

$$
\begin{gathered}
\dot{c}\left\{A_{1}\right\}=A_{1} \\
\dot{c}\left\{A_{2}, A_{3}\right\}=f\left(A_{2}, A_{3}\right\}=\left\{\left\langle x_{1},\{0.6258,0.5528,0.5310,0.5817,0.5,0.4756\}\right\rangle,\right. \\
\left.<x_{2},\{0.7879,0.7551,0.7,0.6536,0.6\}\right\rangle,\left\langle x_{3},\{1\}\right\rangle,
\end{gathered}
$$

$$
\begin{aligned}
& <x_{4},\{0.2789,0.2567,0.2351,0.2,0.1754,0.1515,0.15,0.1254\}>, \\
& \quad<x_{5},\{0.2517,0.2,0.2063,0.1515,0.1633,0.1056\}>, \\
& \quad<x_{6},\{0.8268,0.7879,0.7551,0.8,0.7172,0.7,0.6536\}>, \\
& \quad< \\
& \quad x_{7},\{0.5101,0.4950,0.4343,0.4169,0.3675,0.3481\}>, \\
& \left.<x_{8},\{0.6394,0.6258,0.6,0.5584,0.5417,0.5101,0.5230,0.5050,0.4708\}>\right\} \\
& \dot{c}\left\{A_{4}, A_{5}\right\}=f\left(A_{4}, A_{5}\right\} \\
& =\left\{<x_{1},\{1\}>,<x_{2},\{1\}>,<x_{3},\{0.9,0.8586,0.8775,0.85,0.8268\}>,\right. \\
& <x_{4},\{0.8268,0.8064,0.7879,0.8,0.7551,0.7764,0.75,0.7261\}>, \\
& <x_{5},\{0.6536,0.4343,0.3675\}>,<x_{6},\{0.2567,0.2286\}>, \\
& <x_{7},\{0.2754,0.2517,0.2286,0.2254,0.2,0.1754,0.1784,0.1515,0.1254\}>, \\
& \left.<x_{8},\{0.5417,0.4084,0.3519\}>\right\}
\end{aligned}
$$

Subsequently, we compare each cluster with the other two clusters by Eq.(1.12):

$$
\begin{gathered}
d_{12}\left(\dot{c}\left\{A_{1}\right\}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{2}, A_{3}\right\}, \dot{c}\left\{A_{1}\right\}\right)=0.4283 \\
d_{12}\left(\dot{c}\left\{A_{1}\right\}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{4}, A_{5}\right\}, \dot{c}\left\{A_{1}\right\}\right)=0.3440 \\
d_{12}\left(\dot{c}\left\{A_{2}, A_{3}\right\}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=d_{12}\left(\dot{c}\left\{A_{4}, A_{5}\right\}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.3411
\end{gathered}
$$

Then the HFSs $A_{j}(j=1,2, \cdots, 5)$ can be clustered into the following two clusters: $\left\{A_{1}\right\}$ and $\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}$.

Finally, the above two clusters are further clustered into a unique cluster: $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$.

If we carry out K -means clustering, then we should choose the results of hierarchical clustering as initial clusters. Since the results at $K=1$ and $K=5$ are unique, we shall illustrate Algorithm 2.5 with $K=2,3,4$.
(1) $K=4$ : Using the result obtained from hierarchical clustering $\left\{A_{1}\right\},\left\{A_{2}\right\}$, $\left\{A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$ as the initial cluster to compute the centroids and distances.

$$
\begin{gathered}
\dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}\right\}=A_{2}, \dot{c}\left\{A_{3}\right\}=A_{3}, \dot{c}\left\{A_{4}, A_{5}\right\}=f\left(A_{4}, A_{5}\right) \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{1}\right\}\right)=0, \quad d_{12}\left(A_{1}, \dot{c}\left\{A_{2}\right\}\right)=0.4335 \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{3}\right\}\right)=0.4598, \quad d_{12}\left(A_{1}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3450 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{1}\right\}\right)=0.4335, \quad d_{12}\left(A_{2}, \dot{c}\left\{A_{2}\right\}\right)=0 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{3}\right\}\right)=0.1643, \quad d_{12}\left(A_{2}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3889 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{1}\right\}\right)=0.4598, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{2}\right\}\right)=0.1643 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{3}\right\}\right)=0, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3211 \\
d_{12}\left(A_{4}, \dot{c}\left\{A_{1}\right\}\right)=0.3827, \quad d_{12}\left(A_{4}, \dot{c}\left\{A_{2}\right\}\right)=0.3900 \\
d_{12}\left(A_{4}, \dot{c}\left\{A_{3}\right\}\right)=0.3038, \quad d_{12}\left(A_{4}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.08209 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{1}\right\}\right)=0.3494, \quad d_{12}\left(A_{5}, \dot{c}\left\{A_{2}\right\}\right)=0.3682 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{3}\right\}\right)=0.3132, \quad d_{12}\left(A_{5}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.07412
\end{gathered}
$$

Based on the above distances, we get the classifications: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$. Since the center of each cluster is not changed, the iteration stops.
(2) $K=3$ : Taking $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$ as the initial cluster, then the corresponding results are:

$$
\begin{gathered}
\dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}, A_{3}\right\}=f\left(A_{2}, A_{3}\right), \dot{c}\left\{A_{4}, A_{5}\right\}=f\left(A_{4}, A_{5}\right) \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{1}\right\}\right)=0, \quad d_{12}\left(A_{1}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.4283 \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3450, d_{12}\left(A_{2}, \dot{c}\left\{A_{1}\right\}\right)=0.4335 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.06408, d_{12}\left(A_{2}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3889 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{1}\right\}\right)=0.4598, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.1260 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.3211, d_{12}\left(A_{4}, \dot{c}\left\{A_{1}\right\}\right)=0.3827 \\
d_{12}\left(A_{4}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.3478, \quad d_{12}\left(A_{4}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.08209 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{1}\right\}\right)=0.3494, \quad d_{12}\left(A_{5}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.3193 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{4}, A_{5}\right\}\right)=0.07412
\end{gathered}
$$

and thus, we get the classifications: $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{4}, A_{5}\right\}$. The center of each cluster remains, we finish the iteration.
(3) $K=2$ : Using $\left\{A_{1}\right\}$ and $\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}$ as the initial cluster, and for this case, the results are presented as follows:

$$
\begin{gathered}
\dot{c}\left\{A_{1}\right\}=A_{1}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}=f\left(A_{2}, A_{3}, A_{4}, A_{5}\right) \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{1}\right\}\right)=0, d_{12}\left(A_{1}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}\right)=0.4115 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{1}\right\}\right)=0.4335, \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}\right)=0.2555 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{1}\right\}\right)=0.4598, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}\right)=0.2612
\end{gathered}
$$

$$
\begin{aligned}
& d_{12}\left(A_{4}, \dot{c}\left\{A_{1}\right\}\right)=0.3827, \quad d\left(A_{4}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}\right)=0.1634 \\
& d_{12}\left(A_{5}, \dot{c}\left\{A_{1}\right\}\right)=0.3494, \quad d\left(A_{5}, \dot{c}\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}\right)=0.1396
\end{aligned}
$$

Obviously, the classifications are $\left\{A_{1}\right\}$ and $\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}$. Again the center of each cluster is not changed, so the iterative calculations are completed.

The above example indicates that by taking the results provided by hierarchical clustering as initial cluster, it reduces the iterative number. That is to say, it can substantially raise the iterative efficiency of K-means clustering as compared to the case of randomly initial values. This favors to get the ideal clustering results quickly.

As Torra (2010), Torra and Narukawa (2009) have shown that the envelope of a HFE is just an IFN (Xu 2007a), one can transform the hesitant fuzzy information (Table 2.21) into the intuitionistic fuzzy information shown in Table 2.22 (Chen et al. 2014).

Table 2.22. Intuitionistic fuzzy information

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $(0.2,0.5)$ | $(0.1,0.8)$ | $(0.5,0.3)$ | $(0.9,0)$ |
| $A_{2}$ | $(0.5,0.4)$ | $(0.6,0.15)$ | $(1,0)$ | $(0.15,0.65)$ |
| $A_{3}$ | $(0.45,0.35)$ | $(0.6,0.3)$ | $(0.9,0)$ | $(0.1,0.8)$ |
| $A_{4}$ | $(1,0)$ | $(1,0)$ | $(0.85,0.1)$ | $(0.75,0.15)$ |
| $A_{5}$ | $(0.9,0)$ | $(0.9,0.1)$ | $(0.8,0.1)$ | $(0.7,0.2)$ |
|  | $x_{5}$ | $x_{6}$ | $x_{7}$ | $x_{8}$ |
| $A_{1}$ | $(0.4,0.35)$ | $(0.1,0.9)$ | $(0.3,0.5)$ | $(1,0)$ |
| $A_{2}$ | $(0,0.8)$ | $(0.7,0.15)$ | $(0.5,0.3)$ | $(0.65,0.2)$ |
| $A_{3}$ | $(0.2,0.7)$ | $(0.6,0.2)$ | $(0.15,0.8)$ | $(0.2,0.65)$ |
| $A_{4}$ | $(0.2,0.8)$ | $(0.15,0.85)$ | $(0.1,0.7)$ | $(0.3,0.7)$ |
| $A_{5}$ | $(0.5,0.15)$ | $(0.3,0.65)$ | $(0.15,0.75)$ | $(0.4,0.3)$ |

It is worth pointing out here that Xu (2009a) has clustered for the data of Table 2.22 using intuitionistic fuzzy clustering method, whose results are presented in Table 2.23 (Chen et al. 2014).

Table 2.23. Comparison of two different types of cluster methods

| Classes | Hierarchical hesitant fuzzy <br> K-means clustering algorithm | Intuitionistic fuzzy hierarchical <br> clustering algorithm |
| :---: | :---: | :---: |
| 5 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ |
| 4 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ |
| 3 | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}, A_{5}\right\}$ |
| 2 | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}\right\}$ |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ |

We can see from Table 2.23 that the clustering results of HFSs, to a large extent, agree with those of IFSs, as the envelope of HFE is just an IFN.
Example 2.16 (Chen et al. 2014). The information on six aspects of five tourism resources is evaluated. The six aspects are scale, environmental conditions, integrity, service, tour routes and convenient traffic, which are represented by the HFSs $\quad A_{i}(i=1,2, \cdots, 5) \quad$ in the feature space $X=\left\{x_{1}, x_{2}, \cdots, x_{6}\right\}$. $w=\left(\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6}\right)^{\mathrm{T}}$ is the weight vector of $x_{i}(i=1,2, \cdots, 6)$. The data are listed in Table 2.24 (Chen et al. 2014).

Table 2.24. Hesitant fuzzy information

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.3 .0 .5\}$ | $\{0.6,0.8,0.9\}$ | $\{0.4,0.7\}$ | $\{0.8,0.9\}$ | $\{0.1,0.2,0.4\}$ | $\{0.5,0.6\}$ |
| $A_{2}$ | $\{0.6,0.7\}$ | $\{0.5,0.6,0.8\}$ | $\{0.6,0.8,0.9\}$ | $\{07,0.9\}$ | $\{0.3,0.4\}$ | $\{0.4,0.7\}$ |
| $A_{3}$ | $\{0.4,0.6\}$ | $\{0.8,0.9\}$ | $\{0.5,0.9\}$ | $\{0.6,0.7,0.8\}$ | $\{0.4,0.5\}$ | $\{0.3,0.8\}$ |
| $A_{4}$ | $\{0.2,0.6\}$ | $\{0.4,0.5,0.9\}$ | $\{0.9,1\}$ | $\{0.8,0.9\}$ | $\{0.2,0.5\}$ | $\{0.7,0.9\}$ |
| $A_{5}$ | $\{0.5,0.8\}$ | $\{0.3,0.4\}$ | $\{0.6,0.7\}$ | $\{0.7,0.9\}$ | $\{0.6,0.8\}$ | $\{0.5,0.7\}$ |

Tourism department divides the five scenic areas into three categories.
Step 1. At this step, each of the HFSs $A_{i}(i=1,2, \cdots, 5)$ is considered as a unique cluster: $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$.
Step 2. Compare each of the HFSs $A_{i}(i=1,2, \cdots, 5)$ with all the other four HFSs by using Eq.(1.12). We find that $d_{12}\left(A_{2}, A_{3}\right)=\min \left\{d_{12}\left(A_{i}, A_{j}\right)\right\}$, $i, j=1,2, \cdots, 5$ and $i \neq j$. Considering that only two clusters can be jointed in each stage, the HFSs $A_{j}(j=1,2, \cdots, 5)$ can be clustered into the following four clusters at the second stage: $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$.
Step 3. Calculate the center of each cluster by using Eq.(1.33), and then compare each cluster with the other three clusters by using Eq.(1.12). Subsequently, the HFSs $A_{i}(i=1,2, \cdots, 5)$ can be clustered into the following three clusters at the third stage:

$$
\left\{A_{1}, A_{4}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{5}\right\}
$$

Step 4. Select $\left\{A_{1}, A_{4}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{5}\right\}$ as initial cluster of K-means, and calculate the centroids of all clusters and their distances to each set:

$$
\begin{gathered}
\dot{c}\left\{A_{2}, A_{3}\right\}=f\left(A_{2}, A_{3}\right)=\frac{1}{2}\left(A_{2} \oplus A_{3}\right) \\
\dot{c}\left\{A_{1}, A_{4}\right\}=f\left(A_{1}, A_{4}\right)=\frac{1}{2}\left(A_{1} \oplus A_{4}\right), \dot{c}\left\{A_{5}\right\}=A_{5} \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.1697, \quad d_{12}\left(A_{1}, \dot{c}\left\{A_{1}, A_{4}\right\}\right)=0.1469 \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{5}\right\}\right)=0.2194, d_{12}\left(A_{2}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.0977 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{1}, A_{4}\right\}\right)=0.1457, \quad d_{12}\left(A_{2}, \dot{c}\left\{A_{5}\right\}\right)=0.1556 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.1163, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{1}, A_{4}\right\}\right)=0.1598 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{5}\right\}\right)=0.2111, d_{12}\left(A_{4}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.1816 \\
d_{12}\left(A_{4}, \dot{c}\left\{A_{1}, A_{4}\right\}\right)=0.1070, \quad d_{12}\left(A_{4}, \dot{c}\left\{A_{5}\right\}\right)=0.2361
\end{gathered}
$$

$$
\begin{gathered}
d_{12}\left(A_{5}, \dot{c}\left\{A_{2}, A_{3}\right\}\right)=0.1832, \quad d_{12}\left(A_{5}, \dot{c}\left\{A_{1}, A_{4}\right\}\right)=0.2300 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{5}\right\}\right)=0
\end{gathered}
$$

The new clusters obtained from the above distances are $\left\{A_{1}, A_{4}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{5}\right\}$. Obviously, the center of clusters is not changed, the iterative process stops.

To illustrate the effectiveness and stability of the hierarchical K-means clustering methods, we make a simple test below:

Let $K=3$, instead of selecting hierarchical clustering results as the initial classification, we randomly select $\left\{A_{1}, A_{2}, A_{3}\right\},\left\{A_{4}\right\}$ and $\left\{A_{5}\right\}$ as initial clusters, whose centroids are:

$$
\begin{gathered}
\dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}=f\left(A_{1}, A_{2}, A_{3}\right)=\frac{1}{3}\left(A_{1} \oplus A_{2} \oplus A_{3}\right) \\
\dot{c}\left\{A_{4}\right\}=A_{4}, \quad \dot{c}\left\{A_{5}\right\}=A_{5}
\end{gathered}
$$

The distances between each set and $\dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}, \dot{c}\left\{A_{4}\right\}$ and $\dot{c}\left\{A_{5}\right\}$ are:

$$
\begin{gathered}
d_{12}\left(A_{1}, \dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}\right)=0.1656, \quad d_{12}\left(A_{1}, \dot{c}\left\{A_{4}\right\}\right)=0.1639 \\
d_{12}\left(A_{1}, \dot{c}\left\{A_{5}\right\}\right)=0.2194, \quad d_{12}\left(A_{2}, \dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}\right)=0.1170 \\
d_{12}\left(A_{2}, \dot{c}\left\{A_{4}\right\}\right)=0.1528, \quad d_{12}\left(A_{2}, \dot{c}\left\{A_{5}\right\}\right)=0.1556 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}\right)=0.1354, \quad d_{12}\left(A_{3}, \dot{c}\left\{A_{4}\right\}\right)=0.1778 \\
d_{12}\left(A_{3}, \dot{c}\left\{A_{5}\right\}\right)=0.2111, d_{12}\left(A_{4}, \dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}\right)=0.1889 \\
d_{12}\left(A_{4}, \dot{c}\left\{A_{4}\right\}\right)=0, \quad d_{12}\left(A_{4}, \dot{c}\left\{A_{5}\right\}\right)=0.2361 \\
d_{12}\left(A_{5}, \dot{c}\left\{A_{1}, A_{2}, A_{3}\right\}\right)=0.1806, \quad d_{12}\left(A_{5}, \dot{c}\left\{A_{4}\right\}\right)=0.2361
\end{gathered}
$$

$$
d_{12}\left(A_{5}, \dot{c}\left\{A_{5}\right\}\right)=0
$$

Examining the above distances, one can see except for $A_{1}$, which belongs to the second cluster, other sets are still the same as the initial cluster. Consequently, the classifications become $\left\{A_{1}, A_{4}\right\},\left\{A_{2}, A_{3}\right\}$ and $\left\{A_{5}\right\}$. This result means that we again return to Step 4. The experiment clearly shows that using the results of hierarchical clustering as initial cluster in K-means algorithm is more efficient than randomly choosing initial cluster, i.e., less iteration. Besides, the initial choice does not affect the prediction of K-means clustering, indicating that the presented clustering method is stable.

We can also transform hesitant fuzzy information (i.e. Example 2.16) into intuitionistic fuzzy information through Definition 1.6. Table 2.25 (Chen et al. 2014) compares the propose method with Zhang et al. (2007)'s method.

Table 2.25. Comparison of tourism scenic classification

| Classes | The presented method | Zhang et al. (2007)'s method |
| :---: | :---: | :---: |
| 5 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ |
| 4 | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ |  |
| 3 | $\left\{A_{1}, A_{4}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{5}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$ |
| 2 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\},\left\{A_{5}\right\}$ |  |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\right\}$ |

From Table 2.25, we can see that the two methods with the class 3 are a little different, and other cases are completely the same. This is because the proposed method takes into account the hesitant factors. From the view of mathematics, it is because as an interval-value, the IFN contains the HFE (which is a discrete value) through Torra (2010)'s definition of envelope. Appearance of the difference in certain circumstances is caused by the difference in the data types and in the distribution of discrete values. This demonstrates the importance of the clustering methods for HFSs. The results suggest that when one performs clustering for discrete hesitation fuzzy data, the clustering method for HFSs should be applied, and it is not accurate to handle the data in the form of IFSs.

### 2.7 MST Clustering Algorithm for HFSs

Based on the desirable characters of graph theory, Zhang and Xu (2012) developed a hesitant fuzzy minimal spanning tree (HFMST) clustering algorithm to deal with hesitant fuzzy information.

In what follows, we first define the concept of hesitant fuzzy distance matrix:
Definition 2.16 (Zhang and Xu 2012). Let $A_{j}(j=1,2, \cdots, n)$ be $n$ HFSs, then $Z=\left(z_{i j}\right)_{n \times n}$ is called a hesitant fuzzy distance matrix, where $z_{i j}=z\left(A_{i}, A_{j}\right)$ is the distance between $A_{i}$ and $A_{j}$, which has the following properties:
(1) $0 \leq z_{i j} \leq 1$, for all $i, j=1,2, \cdots, n$.
(2) $z_{i j}=0$ if and only if $A_{i}=A_{j}$.
(3) $z_{i j}=z_{j i}$, for all $i, j=1,2, \cdots, n$.

### 2.7.1 Graph and Minimal Spanning Trees

A graph $\Gamma$ is a pair of sets $\Gamma=(V, \Lambda)$, where $V$ is the set of nodes and $\Lambda$ is the set of edges. In an undirected graph, each edge is an unordered pair $\left\{v_{1}, v_{2}\right\}$. In a directed graph (also called a digraph in some literature), the edges are ordered pairs. The nodes $\nu_{1}$ and $\nu_{2}$ are called the endpoints of an edge. In a weighted graph, $\omega$ is defined as a weight on each edge (Schaeffer 2007). It needs to mention that the graph in the rest of the subsection is the undirected graph. Next, we introduce some other notions of graph by Fig. 2.4 (Zhang and Xu 2012), in which Fig. 2.4(a) depicts a weighted graph with six nodes and nine edges:


Fig.2.4. The graph and the minimal spanning trees

(c) Minimal spanning tree

Fig.2.4. (continued)
A sequence of edges and nodes that can be traveled to go from one node to that of another is called a path. For instance, it might be the case that two different paths exist from the node $A$ to the node $H$, such as the one denoted by ( $A B C F H$ ) and the other one denoted by $(A B C D H)$. If a path where the start node and destination node are the same is called a circuit as $(A B C A)$ or $(A C F H D A)$. A connected graph has paths between any pair of nodes. A connected acyclic graph that contains all nodes of $\Gamma$ is called a spanning tree of the graph. Obviously, Fig. 2.4(b) is one of such graphs. If we define the weight of a tree to be the sum of the weights of its constituent edges, then a minimal spanning tree of the graph $\Gamma$ is a spanning tree, whose weight is minimal among all spanning trees of $\Gamma$ as Fig. 2.4(c) (Zahn 1971).

In fact, the set $\Lambda$ in a normal graph is a crisp relation over $V \times V$. That is to say, if there exists an edge between the nodes $\nu_{1}$ and $\nu_{2}$, then the membership degree equals 1 , i.e., $\mu_{\Lambda}\left(v_{1}, v_{2}\right)=1$; Otherwise $\mu_{\Lambda}\left(v_{1}, v_{2}\right)=0$, where $\left(v_{1}, v_{2}\right) \in V \times V$. If a fuzzy relation $R$ over $V \times V$ is defined, then the membership function $\mu_{R}\left(v_{1}, v_{2}\right)$ takes various values from 0 to 1 , and such a graph is called a fuzzy graph. If $R$ is a hesitant fuzzy relation over $V \times V$, then $\Gamma=(V, R)$ is called a hesitant fuzzy graph.

Based on the hesitant fuzzy distance matrix given in Definition 2.16, Zhang and Xu (2012) used the idea of Zahn (1971) to develop a hesitant fuzzy minimal spanning tree (HFMST) clustering algorithm.

### 2.7.2 HFMST Clustering Algorithm

Let $X=\left\{x_{1}, x_{2}, \cdots, x_{m}\right\}$ be an attribution space and $w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}}$ the weight vector of the elements $x_{j}(j=1,2, \cdots, m)$, with $w_{j} \geq 0$,
$j=1,2, \cdots, m$, and $\sum_{j=1}^{m} w_{j}=1$. Let $A_{i}(i=1,2, \cdots, n)$ be a collection of $n$ HFSs expressing $n$ samples to be clustered, having the following forms:

$$
\begin{equation*}
A_{i}=\left\{<x_{j}, h_{A_{i}}\left(x_{j}\right)>\mid x_{j} \in X\right\}, \quad i=1,2, \cdots, n \tag{2.188}
\end{equation*}
$$

Then we propose a hesitant fuzzy minimal spanning tree (HFMST) clustering algorithm, whose steps are as follows (Zhang and Xu 2012):
(Algorithm 2.6) (HFMST clustering algorithm)
Step 1. Compute the hesitant fuzzy distance matrix and the fuzzy graph:
(1) Calculate the distance $z_{i j}=z\left(A_{i}, A_{j}\right)$ by Eqs.(2.10)-(2.18) and get the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{n \times n}$.
(2) Build the hesitant fuzzy graph $\Gamma=(V, \Lambda)$ where every edge between $A_{i}$ and $A_{j}$ has the weight $z_{i j}$ represented by HFSs as an element of the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{n \times n}$, which shows the dissimilarity degree between the samples $A_{i}$ and $A_{j}$.

Step 2. Compute the MST of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$ by Kruskal's method (Kruskal 1965) (or Prim's method (Prim 1957)):
(1) Sort the edges of $\Gamma$ in increasing order by weight.
(2) Keep a sub-graph $\bar{\Gamma}$ of $\Gamma$, which is initially empty, and choose at each step the edge $\bar{e}$ with the smallest weight to add to $\bar{\Gamma}$, in which the endpoints of $\bar{e}$ are disconnected.
(3) Repeat the process (2) until the sub-graph $\bar{\Gamma}$ spans all nodes. Thus, we get the MST of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$.

Step 3. Make clustering analysis by using the minimal hesitant fuzzy spanning tree. Thus, we can get a certain number of sub-trees (clusters) by disconnecting all the edges of the MST with weights greater than a threshold $\lambda_{0}$. The clustering results induced by the sub-trees do not depend on some particular MST (Gaertler 2002).

### 2.7.3 Numerical Examples

In this subsection, two illustrative examples will be given in order to demonstrate the practical usage and the effectiveness of Algorithm 2.6.

Example 2.17 (Zhang and Xu 2012). Jiangxi province is located in southeast of China and the middle reaches of the Changjiang (Yangtze) River, which enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. However, there are also some restrictive factors for developing agriculture such as a tight man-land relation between, a constant degradation of natural resources and a growing population pressure on land resource reserve. Based on the distinctness and differences in environment and natural resources, Jiangxi Province can be roughly divided into ten cities: $A_{1}$ - Fuzhou, $A_{2}$ - Nanchang, $A_{3}$ - Shangrao, $A_{4}$ - Jiujiang, $A_{5}$ - Pingxiang, $A_{6}-$ Yingtan, $A_{7}$ - Ganzhou, $A_{8}$ - Yichun, $A_{9}$ - Jingdezhen, $A_{10}$ - Ji'an. Hence, in order to co-ordinate the development and improve people's living standards, the local government intends to classify these cities into different regions. Suppose that several DMs are invited to evaluate the performances of the ten alternatives (cities) based on two attributes: (1) $x_{1}$ : Ecological benefit;
$x_{2}$ : Economic benefit. For an alternative under an attribute, although all the DMs provide their evaluated values by using HFEs. The results evaluated by the DMs are described as follows:

$$
\begin{aligned}
& A_{1}=\left\{<x_{1},\{0.8,0.7,0.6\}>,<x_{2},\{0.8,0.7,0.3\}>\right\} \\
& A_{2}=\left\{<x_{1},\{0.9,0.8,0.3\}>,<x_{2},\{0.8,0.7,0.6\}>\right\} \\
& A_{3}=\left\{<x_{1},\{0.9,0.7,0.1\}>,<x_{2},\{0.8,0.7,0.6\}>\right\} \\
& A_{4}=\left\{<x_{1},\{0.9,0.8,0.3\}>,<x_{2},\{0.9,0.8,0.2\}>\right\} \\
& A_{5}=\left\{<x_{1},\{0.8,0.5,0.4\}>,<x_{2},\{0.7,0.6,0.5\}>\right\} \\
& A_{6}=\left\{<x_{1},\{0.9,0.8,0.2\}>,<x_{2},\{0.9,0.8,0.7\}>\right\} \\
& A_{7}=\left\{<x_{1},\{0.8,0.7,0.6\}>,<x_{2},\{0.9,0.7,0.6\}>\right\} \\
& A_{8}=\left\{<x_{1},\{0.9,0.8,0.7\}>,<x_{2},\{0.9,0.8,0.3\}>\right\}
\end{aligned}
$$

$$
\begin{aligned}
& A_{9}=\left\{\left\langle x_{1},\{0.9,0.7,0.3\}\right\rangle,\left\langle x_{2},\{0.9,0.7,0.6\}\right\rangle\right\} \\
& A_{10}=\left\{\left\langle x_{1},\{0.7,0.6,0.5\}\right\rangle,\left\langle x_{2},\{0.9,0.8,0.1\}\right\rangle\right\}
\end{aligned}
$$

Let the weight vector of the attributes $x_{j}(j=1,2)$ be $w=(0.45,0.55)^{\mathrm{T}}$. We utilize the HFMST clustering algorithm to group these operational plans $A_{i}(i=1,2, \cdots, 10)$ :

Step 1. Construct the hesitant fuzzy distance matrix and the fuzzy graph where each node is associated to a city to be clustered which is expressed by a HFS:
(1) Calculate the distance $z_{i j}=z\left(A_{i}, A_{j}\right)$ by Eq.(2.18), and then get the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{10 \times 10}$ :

$$
Z=\left(\begin{array}{cccccccccc}
0 & 0.3502 & 0.4399 & 0.2272 & 0.2502 & 0.469 & 0.2107 & 0.134 & 0.3512 & 0.2035 \\
0.3502 & 0 & 0.0952 & 0.2814 & 0.2272 & 0.131 & 0.2117 & 0.3983 & 0.1155 & 0.463 \\
0.4399 & 0.0952 & 0 & 0.3766 & 0.2327 & 0.1366 & 0.3014 & 0.4916 & 0.1617 & 0.5417 \\
0.2272 & 0.2814 & 0.3766 & 0 & 0.3722 & 0.276 & 0.4217 & 0.2541 & 0.3255 & 0.1824 \\
0.2502 & 0.2272 & 0.2327 & 0.3722 & 0 & 0.3176 & 0.2501 & 0.3263 & 0.2449 & 0.3505 \\
0.469 & 0.131 & 0.1366 & 0.276 & 0.3176 & 0 & 0.2651 & 0.5081 & 0.1291 & 0.5689 \\
0.2107 & 0.2117 & 0.3014 & 0.4217 & 0.2501 & 0.2651 & 0 & 0.3176 & 0.1405 & 0.4047 \\
0.134 & 0.3983 & 0.4916 & 0.2541 & 0.3263 & 0.5081 & 0.3176 & 0 & 0.3969 & 0.2517 \\
0.3512 & 0.1155 & 0.1617 & 0.3255 & 0.2449 & 0.1291 & 0.1405 & 0.3969 & 0 & 0.45 \\
0.2035 & 0.463 & 0.5417 & 0.1824 & 0.3505 & 0.5689 & 0.4047 & 0.2517 & 0.45 & 0
\end{array}\right)
$$

(2) Construct the fuzzy graph $\Gamma=(V, \Lambda)$ where every edge between $A_{i}$ and $A_{j}$ has the weight $z_{i j}$ represented by a HFS as an element of the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{10 \times 10}$, which shows the dissimilarity degree between the samples $A_{i}$ and $A_{j}$ (see Fig. 2.5 (Zhang and Xu 2012)).


Fig. 2.5. The hesitant fuzzy graph $\Gamma=(V, \Lambda)$

Step 2. Compute the MST of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$ by Kruskal's method (Kruskal 1956):
(1) Sort the edges of $\Gamma$ in increasing order by weights:
$z_{23}<z_{29}<z_{26}<z_{18}<z_{36}<z_{79}<z_{39}<z_{4,10}<z_{1,10}<z_{27}<z_{25}=z_{14}<z_{34}$
$<z_{59}<z_{57}<z_{15}<z_{8,10}<z_{4,8}<z_{67}<z_{45}<z_{24}<z_{37}<z_{56}=z_{78}<z_{49}<z_{58}$
$<z_{12}<z_{5,10}<z_{19}<z_{45}<z_{34}<z_{89}<z_{7,10}<z_{47}<z_{13}<z_{9,10}<z_{2,10}<z_{16}$
$<z_{38}<z_{68}<z_{3,10}<z_{6,10}$
(2) Keep an empty sub-graph $\bar{\Gamma}$ of $\Gamma$, and choose the edge $\bar{e}$ with the smallest weight to add to $\bar{\Gamma}$, in which the endpoints of $\bar{e}$ are disconnected, so we can choose the edge $\bar{e}_{23}$ between $A_{2}$ and $A_{3}$.
(3) Repeat the process (2) until the sub-graph $\bar{\Gamma}$ spans ten nodes. Thus, we get the MST of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$ (see Fig. 2.6(a-j) (Zhang and Xu 2012)):


Fig.2.6. The sub-trees of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$


Fig 2.6. (continued)
$A_{10}$
$A_{1}{ }^{\bullet}$


- $\quad A_{6}$
$A_{5}$
(g)

(h)


$A_{5}$
(i)

Fig 2.6. (continued)

(j)

Fig. 2.6. (continued)
Step 3. Select a threshold $\lambda_{0}$ and disconnect all the edges of the MST with weights greater than $\lambda_{0}$ so that we could get a certain number of sub-trees (clusters) automatically, listed in Table 2.26 (Zhang and Xu 2012):

Table 2.26. Clustering results with various values of the threshold $\boldsymbol{\lambda}_{0}$

| $\lambda_{0}$ | Corresponding clustering results | Corresponding <br> MST |
| :---: | :---: | :---: |
| $\lambda_{0}=z_{34}=0.3766$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ | Fig. 2.6(a) |
| $\lambda_{0}=z_{14}=z_{25}=0.2272$ | $\left\{A_{1}, A_{4}, A_{8}, A_{10}\right\},\left\{A_{2}, A_{3}, A_{5}, A_{6}, A_{7}, A_{9}\right\}$ | Fig. 2.6(b) |
| $\lambda_{0}=z_{1,10}=0.2305$ | $\left\{A_{5}\right\},\left\{A_{1}, A_{4}, A_{8}, A_{10}\right\},\left\{A_{2}, A_{3}, A_{6}, A_{7}, A_{9}\right\}$ | Fig. 2.6(c) |
| $\lambda_{0}=z_{4,10}=0.1824$ | $\left\{A_{5}\right\},\left\{A_{1}, A_{8}\right\},\left\{A_{4}, A_{10}\right\},\left\{A_{2}, A_{3}, A_{6}, A_{7}, A_{9}\right\}$ | Fig. 2.6(d) |
| $\lambda_{0}=z_{79}=0.1405$ | $\left\{A_{5}\right\},\left\{A_{1}, A_{8}\right\},\left\{A_{4}\right\},\left\{A_{10}\right\},\left\{A_{2}, A_{3}, A_{6}, A_{7}, A_{9}\right\}$ | Fig. 2.6(e) |

Table 2.26. (continued)

| $\lambda_{0}=z_{18}=0.134$ | $\left\{A_{1}, A_{8}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{7}\right\},\left\{A_{10}\right\},\left\{A_{2}, A_{3}, A_{6}, A_{9}\right\}$ | Fig. 2.6(f) |
| :---: | :---: | :---: |
| $\lambda_{0}=z_{26}=0.131$ | $\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{10}\right\},\left\{A_{2}, A_{3}, A_{6}, A_{9}\right\}$ | Fig. 2.6(g) |
| $\lambda_{0}=z_{29}=0.1155$ | $\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{10}\right\},\left\{A_{2}, A_{3}, A_{9}\right\}$ | Fig. 2.6(h) |
| $\lambda_{0}=z_{23}=0.0952$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ | Fig. 2.6(i) |
| $\lambda_{0}=0$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ | Fig. 2.6(i) |

Obviously, according to Table 2.26 based on real needs, the Jiangxi provincial government can make its ten cities to be divided into different agroecological regions (clusters) in order to improve its overall development. For instance, if the government intends to classify these ten cities into four agroecological regions (clusters), thus, it can easily obtain the results from Table 2.26, which is derived by utilizing Algorithm 2.6 to compute the assessment values of alternatives (cities) provided by the DMs (experts), as follows:

The first agroecological region includes: $A_{5}-$ Pingxiang; the second agroecological region includes: $A_{1}-$ Fuzhou, and $A_{8}-$ Yichun; the third agroecological region includes: $A_{4}$ - Jiujiang and $A_{10}$ - Ji'an; and the fourth agroecological region includes: $A_{2}-$ Nanchang, $A_{3}-$ Shangrao, $A_{5}$ - Pingxiang, $A_{6}$-Yingtan, $A_{7}$ - Ganzhou, and $A_{9}$ - Jingdezhen.

It is noted that the numbers of values in different HFEs of HFSs are the same in Example 2.17. However in most cases, the numbers of values in different HFEs of HFSs may be different. In Example 2.18, we will make further discussion in detail.

To compare with the intuitionistic fuzzy MST (IFMST) clustering algorithm and the fuzzy MST (FMST) clustering algorithm, we give another example with six nodes for convenience. In Example 2.18, we will first make clustering analysis under hesitant fuzzy environment, and then consider the HFSs' envelopes, i.e., intuitionistic fuzzy data, and make an IFMST clustering analysis. Finally, we will make a FMST clustering analysis when the considered intuitionistic fuzzy sets reduce to the fuzzy sets by considering only the membership degrees of the data.

Example 2.18 (Zhang and Xu 2012). In order to complete an operational mission, six sets of operational plans are made initially (adapted from Zhang et al. (2009) and Zhao et al. (2012)). To group these operational plans with respect to their comprehensive functions, a military committee has been set up to provide
assessment information on them. The attributes which are considered here in assessment of the six sets of operational plans are: (1) $x_{1}$ is the effectiveness of operational organization; (2) $x_{2}$ is the effectiveness of operational command. The military committee evaluates the performance of the six operational plans according to the attributes $x_{j}(j=1,2)$, and gives the hesitant fuzzy data as:
$\left.A_{1}=\left\{<x_{1},\{0.85,0.70\}\right\rangle,\left\langle x_{2},\{0.80,0.75,0.60\}\right\rangle\right\}$
$A_{2}=\left\{\left\langle x_{1},\{0.65,0.5,0.4\}\right\rangle,\left\langle x_{2},\{0.9,0.8\}\right\rangle\right\}$
$A_{3}=\left\{\left\langle x_{1},\{0.75,0.6,0.55\}\right\rangle,\left\langle x_{2},\{0.85,0.8,0.7\}\right\rangle\right\}$
$A_{4}=\left\{\left\langle x_{1},\{0.65,0.44\}\right\rangle,\left\langle x_{2},\{0.8,0.7,0.6\}\right\rangle\right\}$
$A_{5}=\left\{\left\langle x_{1},\{0.65,0.6,0.5\}\right\rangle,\left\langle x_{2},\{0.8,0.75\}\right\rangle\right\}$
$A_{6}=\left\{<x_{1},\{0.75,0.6,0.55\}>,<x_{2},\{0.85,0.7,0.57\}>\right\}$

Apparently, the numbers of values in different HFEs of HFSs are different. To operate correctly, we consider that the DMs are pessimistic in Example 2.18, so we change the hesitant fuzzy data by adding the minimal values as below (for convenience of description, here we also list them in the corresponding sets):
$A_{1}=\left\{<x_{1},\{0.85,0.7,0.7\}>,<x_{2},\{0.8,0.75,0.6\}>\right\}$
$A_{2}=\left\{\left\langle x_{1},\{0.65,0.5,0.4\}\right\rangle,\left\langle x_{2},\{0.9,0.8,0.8\}\right\rangle\right\}$
$A_{3}=\left\{\left\langle x_{1},\{0.75,0.6,0.55\}\right\rangle,\left\langle x_{2},\{0.85,0.8,0.7\}\right\rangle\right\}$
$A_{4}=\left\{<x_{1},\{0.65,0.44,0.44\}>,<x_{2},\{0.8,0.7,0.6\}>\right\}$
$\left.A_{5}=\left\{\left\langle x_{1},\{0.65,0.6,0.5\}\right\rangle,<x_{2},\{0.8,0.75,0.75\}\right\rangle\right\}$
$A_{6}=\left\{<x_{1},\{0.75,0.6,0.55\}>,<x_{2},\{0.85,0.7,0.57\}>\right\}$

Then we proceed to utilize the HFMST clustering algorithm to group these operational plans $A_{j}(j=1,2, \cdots, 6)$ :

Step 1. Construct the hesitant fuzzy distance matrix and the hesitant fuzzy graph:
(1) Calculate the distance $z_{i j}=z\left(A_{i}, A_{j}\right)$ by Eq.(2.18), and then we get the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{6 \times 6}$ as:

$$
Z=\left(\begin{array}{cccccc}
0.0000 & 0.3264 & 0.1474 & 0.1733 & 0.2052 & 0.1140 \\
0.3264 & 0.0000 & 0.1480 & 0.1761 & 0.1899 & 0.2406 \\
0.1474 & 0.1480 & 0.0000 & 0.1609 & 0.090 & 0.0965 \\
0.1733 & 0.1761 & 0.6090 & 0.0000 & 0.1540 & 0.1216 \\
0.2052 & 0.1899 & 0.0900 & 0.1540 & 0.0000 & 0.1735 \\
0.1140 & 0.2406 & 0.0965 & 0.1216 & 0.1735 & 0.0000
\end{array}\right)
$$

(2) Construct the fuzzy graph $\Gamma=(V, \Lambda)$ where every edge between $A_{i}$ and $A_{j}$ has the weight $z_{i j}$ represented by HFSs as an element of the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{6 \times 6}$, which shows the dissimilarity degree between the samples $A_{i}$ and $A_{j}$ (see Fig. 2.7) (Zhang and Xu 2012):


Fig.2.7. The hesitant fuzzy graph $\Gamma=(V, \Lambda)$
Step 2. Compute the hesitant fuzzy MST of the hesitant fuzzy graph $\Gamma=(V, \Lambda)$. See Step 2 in the HFMST clustering algorithm.
Step 3. Group the nodes (the operational plans) into clusters. See Step 3 in the HFMST clustering algorithm.

Hence, after the above steps, we obtain the corresponding clustering results, listed in Table 2.27 (Zhang and Xu 2012):

Table 2.27. The HFMST clustering results

| $\lambda_{0}$ | Corresponding clustering results |
| :---: | :---: |
| $\lambda_{0}=z_{23}=0.1474$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{46}=0.1216$ | $\left\{A_{2}\right\},\left\{A_{1}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{16}=0.114$ | $\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{1}, A_{3}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{36}=0.0965$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{3}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{35}=0.09$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{3}, A_{5}\right\}$ |
| $\lambda_{0}=0$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\}$ |

According to Defined 1.6, the IFN $\alpha_{e n v}(h)$ is the envelope of the HFE $h$, then we can transform the hesitant fuzzy data of Example 2.18 into intuitionistic fuzzy data:

$$
\begin{aligned}
& A_{1}=\left\{<x_{1}, 0.70,0.15>,<x_{2}, 0.60,0.20>\right\} \\
& A_{2}=\left\{<x_{1}, 0.40,0.35>,<x_{2}, 0.80,0.10>\right\} \\
& A_{3}=\left\{<x_{1}, 0.55,0.25>,<x_{2}, 0.70,0.15>\right\} \\
& \left.A_{4}=\left\{<x_{1}, 0.44,0.35>,<x_{2}, 0.60,0.20\right\rangle\right\} \\
& \left.A_{5}=\left\{<x_{1}, 0.50,0.35>,<x_{2}, 0.75,0.20\right\rangle\right\} \\
& \left.A_{6}=\left\{<x_{1}, 0.55,0.25>,<x_{2}, 0.57,0.15\right\rangle\right\}
\end{aligned}
$$

and then the attributes $x_{j}(j=1,2, \cdots, 6)$ can be clustered as the following IFMST clustering algorithm (Zhao et al. 2012):

Step 1. Compute the intuitionistic fuzzy distance matrix and the fuzzy graph:
(1) Calculate $z_{i j}=z\left(A_{i}, A_{j}\right)$ by the following distance measure (2.186), where the weight vector of the criteria $x_{j}(j=1,2)$ is $w=(0.45,0.55)^{\mathrm{T}}$, and we get the fuzzy distance matrix:

$$
Z=\left(\begin{array}{cccccc}
0.0000 & 0.2450 & 0.1225 & 0.1170 & 0.1725 & 0.1115 \\
0.2450 & 0.0000 & 0.12250 & 0.1280 & 0.1000 & 0.1940 \\
0.1225 & 0.1225 & 0.0000 & 0.1045 & 0.1000 & 0.0715 \\
0.1170 & 0.1280 & 0.1045 & 0.0000 & 0.1095 & 0.0935 \\
0.1725 & 0.1000 & 0.1000 & 0.1095 & 0.0000 & 0.1715 \\
0.1115 & 0.1940 & 0.0715 & 0.0935 & 0.1715 & 0.0000
\end{array}\right)
$$

(2) Construct the fuzzy graph $\Gamma=(V, \Lambda)$ where every edge between $A_{i}$ and $A_{j}$ has the weight $z_{i j}$ represented by a HFS as an element of the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{6 \times 6}$, which shows the dissimilarity degree between the samples $A_{i}$ and $A_{j}$ (see Fig. 2.7 (Zhang and Xu 2012)).
Step 2. Compute the MST of the intuitionistic fuzzy graph $\Gamma=(V, \Lambda)$. See also Step 2 in the HFMST clustering algorithm.
Step 3. Group the nodes (the operational plans) into clusters. See also Step 3 in the HFMST clustering algorithm.

Obviously, after the above steps, we can obtain the corresponding clustering results, listed in Table 2.28 (Zhang and Xu 2012):

Table 2.28. The IFMST clustering results

| $\lambda_{0}$ | Corresponding clustering results |
| :---: | :---: |
| $\lambda_{0}=z_{16}=0.1115$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{25}=z_{35}=0.1$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{46}=0.088$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{5}\right\},\left\{A_{3}, A_{4}, A_{6}\right\}$ |
| $\lambda_{0}=z_{36}=0.0715$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{3}, A_{6}\right\}$ |
| $\lambda_{0}=0$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\}$ |

It is well known that both the intuitionistic fuzzy set and the HFS are the extensions of the traditional fuzzy set. Compared with the intuitionistic fuzzy set, each of its elements is composed of a membership degree, a non-membership degree and a hesitancy degree, each element of a fuzzy set is only composed of the membership degree. So the intuitionistic fuzzy data are reduced to the fuzzy data when we only consider the membership degrees of the intuitionistic data, then the operational plans' information given by the military committee will be:

$$
\begin{aligned}
& A_{1}=\left\{\left\langle x_{1}, 0.70\right\rangle,\left\langle x_{2}, 0.60\right\rangle\right\}, A_{2}=\left\{\left\langle x_{1}, 0.40\right\rangle,\left\langle x_{2}, 0.80\right\rangle\right\} \\
& A_{3}=\left\{\left\langle x_{1}, 0.55\right\rangle,\left\langle x_{2}, 0.70\right\rangle\right\}, A_{4}=\left\{\left\langle x_{1}, 0.44\right\rangle,\left\langle x_{2}, 0.60\right\rangle\right\} \\
& A_{5}=\left\{\left\langle x_{1}, 0.50\right\rangle,\left\langle x_{2}, 0.75\right\rangle\right\}, A_{6}=\left\{\left\langle x_{1}, 0.55\right\rangle,\left\langle x_{2}, 0.57\right\rangle\right\}
\end{aligned}
$$

and then the operational plans $A_{i}(i=1,2, \cdots, 6)$ can be clustered as the following FMST clustering algorithm:

Step 1. Compute the fuzzy distance matrix and the fuzzy graph:
(1) Calculate $z_{i j}=z\left(A_{i}, A_{j}\right)$ by the following distance measure:

$$
\begin{equation*}
z\left(A_{i}, A_{j}\right)=\sum_{k=1}^{2} w_{k}\left(\left|\mu_{A_{i}}\left(x_{k}\right)-\mu_{A_{j}}\left(x_{k}\right)\right|\right) \tag{2.189}
\end{equation*}
$$

where the weight vector of the attributes $x_{k}(k=1,2)$ is $w=(0.45,0.55)^{\mathrm{T}}$, and we get the fuzzy distance matrix:

$$
D=\left(\begin{array}{llllll}
0.0000 & 0.2450 & 0.1225 & 0.1170 & 0.1725 & 0.0840 \\
0.2450 & 0.0000 & 0.1225 & 0.1280 & 0.0725 & 0.1940 \\
0.1225 & 0.1225 & 0.0000 & 0.1045 & 0.0500 & 0.0715 \\
0.1170 & 0.1280 & 0.1045 & 0.0000 & 0.1095 & 0.0660 \\
0.1725 & 0.0725 & 0.0500 & 0.1095 & 0.0000 & 0.1215 \\
0.0840 & 0.1940 & 0.0715 & 0.0660 & 0.1215 & 0.0000
\end{array}\right)
$$

(2) Construct the fuzzy graph $\Gamma=(V, \Lambda)$ where every edge between $A_{i}$ and $A_{j}$ has the weight $z_{i j}$ represented by a HFS as an element of the hesitant fuzzy distance matrix $Z=\left(z_{i j}\right)_{6 \times 6}$, which shows the dissimilarity degree between the samples $A_{i}$ and $A_{j}$ (see Fig. 2.7).
Step 2. Compute the MST of the fuzzy graph $\Gamma=(V, \Lambda)$. See also Step 2 in the HFMST clustering algorithm.
Step 3. Group the nodes (the operational plans) into clusters. See also Step 3 in the HFMST clustering algorithm.

Analogously, after the above steps, we can get the corresponding clustering results, listed in Table 2.29 (Zhang and Xu 2012):

Table 2.29. The FMST clustering results

| $\lambda_{0}$ | Corresponding clustering results |
| :---: | :---: |
| $\lambda_{0}=z_{16}=0.0840$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{25}=0.0725$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{36}=0.0715$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| $\lambda_{0}=z_{46}=0.0660$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{5}\right\},\left\{A_{4}, A_{6}\right\}$ |
| $\lambda_{0}=z_{35}=0.0500$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{5}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\}$ |
| $\lambda_{0}=0$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\}$ |

In order to provide a better view of the comparison results, we put the clustering results of those three algorithms into Table 2.30 (Zhang and Xu 2012):

Table 2.30. Clustering results

| Classe <br> s | The HFMST clustering Algorithm | The IFMST clustering algorithm | The FMST <br> clustering <br> algorithm |
| :---: | :---: | :---: | :---: |
| 6 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\} \\ \left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\}, \\ \left\{A_{5}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\}, \\ \left\{A_{4}\right\}, \\ \left\{A_{5}\right\},\left\{A_{6}\right\} \\ \hline \end{gathered}$ |
| 5 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{6}\right. \\ \left\{A_{3}, A_{5}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{3}, A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\} \\ , \\ \left\{A_{6}\right\},\left\{A_{3}, A_{5}\right\} \end{gathered}$ |
| 4 | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{4}\right\}, \\ \left\{A_{3}, A_{5}, A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{5}\right\}, \\ \left\{A_{3}, A_{4}, A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}, A_{5}\right\} \\ ,\left\{A_{4}, A_{6}\right\} \end{gathered}$ |
| 3 | $\begin{gathered} \left\{A_{2}\right\},\left\{A_{4}\right\}, \\ \left\{A_{1}, A_{3}, A_{5}, A_{6}\right\} \end{gathered}$ |  | $\begin{gathered} \left\{A_{1}\right\},\left\{A_{2}\right\}, \\ \left\{A_{3}, A_{4}, A_{5}, A_{6}\right\} \end{gathered}$ |
| 2 | $\left\{A_{2}\right\},\left\{A_{1}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ |

After calculations, we find that the clustering results of those three clustering algorithms are quite different. The main reason is that the HFMST clustering algorithm clusters the fuzzy information which is represented by several possible values, not by a margin of error (as in intuitionistic fuzzy sets), while the FMST clustering algorithm clusters the fuzzy information which only considers the membership degrees and thus loses too much information. Obviously, compared with the clustering results of the FMST clustering algorithm, both the HFMST clustering results and the IFMST clustering results are more reasonable. Moreover, when we meet some situations where the information is represented by several possible values, the HFMST clustering algorithm demonstrates its great superiority in clustering those hesitant fuzzy data.

## Chapter 3 <br> Hesitant Preference Relations

Fuzzy preference relations (Orlovsky 1978) (also known as reciprocal preference relation (Baets et al. 2006; Xu 2007b,f; Xu and Chen 2008b)) and multiplicative preference relations (Saaty 1980) are the most common tools to express the DMs' preferences over alternatives in decision making. However, sometimes, to get a more reasonable decision result, a decision organization, which contains a lot of DMs (or experts), is authorized to provide the preferences by comparing each pair of alternatives using $0-1$ scale, and when providing the degrees to which an alternative is superior to another, it is not very sure about a value but has hesitancy between several possible values. In such cases, these several possible values can be considered as a HFE, and a hesitant fuzzy preference relation is constructed when all the preferences over a set of alternatives are provided (Xia and Xu 2013).

It is noted that the hesitant fuzzy preference relation is developed based on the fuzzy preference relation, whose values are expressed by using the $0-1$ scale which is uniformly and symmetrically distributed around 0.5 . But generally speaking, the grades of the preference are not symmetrical but unsymmetrical distributed around some value. Especially, the distances between the grades expressing good information should be bigger than the ones between the grades expressing the bad information in our intuition. Saaty's 1-9 scale is a useful tool to deal with such a situation, especially in expressing a multiplicative preference relation which has been applied in many areas. Although lots of work has provided mechanisms to covert multiplicative preference relation to fuzzy ones and vise-versa, some original information may be lost in the transformation process. Recently, people have been paying more and more attention to personalization service, and therefore decision techniques should be more flexible to satisfy the DMs' different demands. Some DMs may think the $0-1$ scale can express their preferences best, some may think Saaty's 1-9 scale can express their preferences more subjectively. If we transform their preferences by using other scales, the process may distort their original information. On the other hand, when we use the transformation formulas, we have to choose the most suitable one from a lot of transformation formulas before makng decision. Different transformation formulas are developed based on different relationships between these two scales, but it is very hard to recognize these relations in practical decision making problems. Different transformation formulas may produce different transformation results, and thus may produce different
decision results, which makes the decision more complex. Therefore, some techniques should be developed suitable for different scale-based preference relations but do not focus on the transformation formulas. If the DMs in the decision organization don't like to use the values between 0 and 1 but would like to use Saaty's ratio scale (as in multiplicative preference relations) to provide the degree that the alternative $A_{i}$ is superior to $A_{j}$, for example, some DMs in the decision organization provide $\frac{1}{3}$, some provide 1 , and the others provide 5 , then the degrees to which the alternative $A_{i}$ superior to $A_{j}(i \neq j)$ can be represented by $a_{i j}=\left\{\frac{1}{3}, 1,5\right\}$ which are called a HME (Xia and Xu 2011c). If the decision organization provides the preference that the alternative $A_{i}$ is superior to $A_{k}(k \neq i \neq j)$, some of the DMs may provide $\frac{1}{2}$, the others may provide 1 , which can be represented by $b_{i k}=\left\{\frac{1}{2}, 1\right\}$. If the decision organization provides the preference that the alternative $A_{k}$ is superior to $A_{l}(l \neq i \neq j \neq k)$, all the members in the decision organization may agree the value 3 , which can be represented by $b_{k l}=\{3\}$. In such cases, we don't consider the decision organization as three DMs or two DMs or one but as a whole which provides all the possible preference values about a set of alternatives, and construct a hesitant multiplicative preference relation. Xia and Xu (2013) defined the concept of hesitant fuzzy preference relation, based on which they gave an approach to group decision making. They also introduced the hesitant multiplicative preference relation which provides the DMs a very useful tool to express their multiplicative hesitant preferences over alternatives. Liao et al. (2013) gave the concepts of multiplicative consistency, acceptable multiplicative consistency for the hesitant fuzzy preference relation, based on which, two algorithms were given to improve the inconsistency level of a hesitant fuzzy preference relation. Furthermore, the consensus of group decision making based on hesitant fuzzy preference relations was also investigated. Zhu and Xu (2013a) developed two regression methods that transform hesitant fuzzy preference relations into fuzzy preference relations. On the basis of the complete consistency, a reduced fuzzy preference relation can be obtained from a hesitant fuzzy preference relation with the highest consistency level. Based on the weak consistency, another regression method was developed to transform hesitant fuzzy preference relations into "reduced fuzzy preference relations" which all satisfy the weak consistency. Zhu et al. (2013b) put forward two principles to normalize HFEs, i.e., $\alpha$-normalization and $\beta$-normalization, based on which they developed a hesitant
goal programming model to derive priorities from hesitant fuzzy preference relations and some consistency measures of hesitant fuzzy preference relations. Additionally, Zhu and Xu (2013b) developed a hesitant fuzzy programming method to derive priorities from hesitant multiplicative preference relation in AHP-hesitant group decision making. The method provides a group consensus index that measures the satisfaction degree of the group solution, integrates the group synthesis and prioritization together, increases the richness of numerical representation of comparison judgments, and results in a best group solution with the highest satisfaction degree. In this chapter, we will focus on group decision making with hesitant fuzzy preference relations.

### 3.1 Hesitant Fuzzy Preference Relations in Group Decision Making

Let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a set of alternatives, then $U=\left(u_{i j}\right)_{n \times n}$ is called a fuzzy preference relation (Orlovsky 1978) on $A \times A$ with the condition that $u_{i j} \geq 0, u_{i j}+u_{j i}=1, i, j=1,2, \cdots, n$, where $u_{i j}$ denotes the degree that the alternative $A_{i}$ is prior to the alternative $A_{j} . u_{i j}=0.5$ implies indifference between $A_{i}$ and $A_{j}$ denoted by $A_{i} \prec A_{j} ; 0 \leq u_{i j}<0.5$ implies that $A_{j}$ is preferred to $A_{i}$ denoted by $A_{j} \succ A_{i}$, the smaller the value of $u_{i j}$, the stronger the preference of the alternative $A_{j}$ over $A_{i} ; 0.5<u_{i j} \leq 1$ implies that $A_{i}$ is preferred to $A_{j}$ denoted by $A_{i} \succ A_{j}$, the bigger the value of $u_{i j}$, the stronger the preference of the alternative $A_{i}$ over $A_{j}$. It is noted that the value $u_{i j}$ in a fuzzy preference relation is a certain value between 0 and 1 . If a decision organization containing a lot of DMs is authorized to provide the degrees to which $A_{i}$ is preferred to $A_{j}$, some DMs provide $h_{i j}^{1}$, some provide $h_{i j}^{2}$ and the others provide $h_{i j}^{3}$, where $h_{i j}^{3}, h_{i j}^{3}, h_{i j}^{3} \in[0,1]$, then in such a case, the preference information $h_{i j}$ that $A_{i}$ is preferred to $A_{j}(i \neq j)$ can be considered as a HFE $h_{i j}=\left\{h_{i j}^{1}, h_{i j}^{2}, h_{i j}^{3}\right\}$. For the alternatives $A_{i}$ and $A_{k}(k \neq i \neq j)$, some DMs in the decision organization may provide $h_{i k}^{1}$, and the others may provide $h_{i k}^{2}$, then the preference information $h_{i k}$ that $A_{i}$ is preferred to $A_{k}$ can be considered as a HFE $h_{i k}=\left\{h_{i k}^{1}, h_{i k}^{2}\right\}$. In such cases, we can not consider the decision organization as three or two DMs, but just as a whole providing all the possible preference
information about alternatives. All $h_{i j}(i, j=1,2, \cdots, n)$ can construct a hesitant fuzzy preference relation defined as follows:

Definition 3.1 (Xia and Xu 2013). Let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a fixed set, then a hesitant fuzzy preference relation $H$ on $A$ is presented by a matrix $H=\left(h_{i j}\right)_{n \times n} \subset A \times A$, where $h_{i j}=\left\{h_{i j}^{t} \mid t=1,2, \cdots, l_{h_{i j}}\right\}$ is a HFE indicating all the possible degrees to which $A_{i}$ is preferred to $A_{j}$. Moreover, $h_{i j}$ should satisfy the following conditions:

$$
\begin{equation*}
h_{i j}^{\sigma(t)}+h_{j i}^{\sigma\left(l_{h i j}-t+1\right)}=1, h_{i i}=\{0.5\}, l_{h_{i j}}=l_{h_{j i}}, i, j=1,2, \cdots, n \tag{3.1}
\end{equation*}
$$

Consider a group decision making problem, let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a discrete set of alternatives, $D_{k}\left(k=1,2, \cdots, p_{0}\right)$ the set of decision organizations and $v=\left(v_{1}, v_{2}, \cdots, v_{p_{0}}\right)^{\mathrm{T}}$ the weight vector of the decision organizations with $\sum_{k=1}^{p_{0}} v_{k}=1$ and $v_{k} \in[0,1], k=1,2, \cdots, p_{0}$. The decision organization $D_{k}$ provides all the possible preference values for each pair of alternatives, and constructs a hesitant fuzzy preference relation $H^{(k)}=\left(h_{i j}^{(k)}\right)_{n \times n}$. By the above analysis, we can develop an approach to group decision making problem based on the hesitant fuzzy preference relations, which can be described as follows (Xia and Xu 2013):

Step 1. Utilize the GHFA operator in Definition 1.18) (or the GHFG operator in Definition 1.19) to aggregate all $h_{i j}^{(k)}(j=1,2, \cdots, n)$ corresponding to the alternative $A_{i}$, and then get the averaged HFE $h_{i}^{(k)}$ of the alternative $A_{i}$ over all the other alternatives for the decision organization $D_{k}$ :

$$
\begin{equation*}
h_{i}^{(k)}=\operatorname{GHFA}_{\lambda}\left(h_{i 1}^{(k)}, h_{i 2}^{(k)}, \cdots, h_{i n}^{(k)}\right)=\left(\frac{1}{n} \oplus_{j=1}^{n}\left(h_{i j}^{(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \tag{3.2}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i}^{(k)}=\operatorname{GHFG}_{\lambda}\left(h_{i 1}^{(k)}, h_{i 2}^{(k)}, \cdots, h_{i n}^{(k)}\right)=\frac{1}{\lambda}\left({\underset{\bigotimes}{j=1}}_{n}^{j}\left(\lambda h_{i j}^{(k)}\right)^{\frac{1}{n}}\right) \tag{3.3}
\end{equation*}
$$

where $\lambda$ is a positive real number.

Step 2. Utilize the GHFWA (or the GHFWG) operator to aggregate all $h_{i}^{(k)}\left(k=1,2, \cdots, p_{0}\right)$ into a collective HFE $h_{i}$ of the alternative $A_{i}$ over all the other alternatives:

$$
\begin{equation*}
h_{i}=\operatorname{GHWFA}_{\lambda}\left(h_{i}^{(1)}, h_{i}^{(2)}, \cdots, h_{i}^{\left(p_{0}\right)}\right)=\left(\sum_{k=1}^{p_{0}}\left(v_{k}\left(h_{i}^{(k)}\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}} \tag{3.4}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i}=\operatorname{GHWFG}_{\lambda}\left(h_{i}^{(1)}, h_{i}^{(2)}, \cdots, h_{i}^{\left(p_{0}\right)}\right)=\left(\prod_{k=1}^{p_{0}}\left(v_{k}\left(h_{i}^{(k)}\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}} \tag{3.5}
\end{equation*}
$$

where $\lambda$ is a positive real number.

Step 3. Calculate the scores of $h_{i}(i=1,2, \cdots, n)$, and then rank all the alternatives $A_{i}(i=1,2, \cdots, n)$ and select the best one in accordance with the values of $s\left(h_{i}\right)(i=1,2, \cdots, n)$.

Example 3.1 (Bazzazi et al. 2011). Consider a problem of selecting a loadinghauling system for a hypothetical iron ore open pit mine. Three potential transportation systems are evaluated: (1) $A_{1}$ : Shovel-truck system; (2) $A_{2}$ : Shovel- truck-in-pit crusher-belt conveyor system; (3) $A_{3}$ : Loader truck system.

To get more objective results, three decision organizations, $D_{k}(k=1,2,3)$ (whose weight vector is $v=(0.5,0.3,0.2)^{\mathrm{T}}$ ), are authorized to provide their preferences over these three systems $A_{i}(i=1,2,3)$. The three decision organizations compare these three systems and provide their preference values. Take $A_{1}$ as an example, the DMs in $D_{1}$ evaluate the degree to which $A_{1}$ is preferred to $A_{2}$, some DMs provide 0.1 , some provide 0.3 , and the rest give 0.4. However, these three parts in decision organization $D_{1}$ can't persuade each other, then the preference information that $A_{1}$ is preferred to $A_{2}$ provided by the decision organization $D_{1}$ can be considered as a HFE $\{0.1,0.3,0.4\}$ as a whole. In such a case, the other existing generalizations of fuzzy set are invalid. Hence, the decision organizations construct their hesitant fuzzy preference relations, which are listed in Tables 3.1-3.3 (Xia and Xu 2013), respectively.

Table 3.1. The hesitant fuzzy preference relation $H_{1}$

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5\}$ | $\{0.1,0.3,0.4\}$ | $\{0.4,0.6,0.7,0.8\}$ |
| $A_{2}$ | $\{0.6,0.7,0.9\}$ | $\{0.5\}$ | $\{0.5,0.6,0.9\}$ |
| $A_{3}$ | $\{0.2,0.3,0.4,0.6\}$ | $\{0.1,0.4,0.5\}$ | $\{0.5\}$ |

Table 3.2. The hesitant fuzzy preference relation $H_{2}$

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5\}$ | $\{0.2,0.4\}$ | $\{0.5,0.6,0.8\}$ |
| $A_{2}$ | $\{0.6,0.8\}$ | $\{0.5\}$ | $\{0.5,0.6,0.8,0.9\}$ |
| $A_{3}$ | $\{0.2,0.4,0.5\}$ | $\{0.1,0.2,0.4,0.5\}$ | $\{0.5\}$ |

Table 3.3. The hesitant fuzzy preference relation $H_{3}$

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5\}$ | $\{0.3,0.4,0.5,0.7\}$ | $\{0.5,0.8\}$ |
| $A_{2}$ | $\{0.3,0.5,0.6,0.7\}$ | $\{0.5\}$ | $\{0.5,0.8,0.9\}$ |
| $A_{3}$ | $\{0.2,0.5\}$ | $\{0.1,0.2,0.5\}$ | $\{0.5\}$ |

To get the optimal choice, the following steps are given:

Step 1. Utilize the GHFA operator (3.2) (without loss of generality, let $\lambda=1$ ) to aggregate all $h_{i j}^{(k)}(j=1,2,3)$ corresponding to the alternative $A_{i}$, and then get the averaged HFE $h_{i}^{(k)}$ of the alternative $A_{i}$ over all the other alternatives for the decision organization $D_{k}$, for example:

$$
h_{1}^{(2)}=\operatorname{GHFA}\left(h_{13}^{(2)}, h_{23}^{(2)}, h_{33}^{(2)}\right)=\operatorname{HFA}(\{0.5\},\{0.2,0.4\},\{0.5,0.6,0.8\})
$$

$$
\begin{aligned}
&=\left\{1-\prod_{i=1}^{3}((1-0.5) \times(1-0.2) \times(1-0.5))^{\frac{1}{3}}, 1-\prod_{i=1}^{3}((1-0.5) \times(1-0.4) \times(1-0.5))^{\frac{1}{3}}\right. \\
& 1-\prod_{i=1}^{3}((1-0.5) \times(1-0.2) \times(1-0.6))^{\frac{1}{3}}, 1-\prod_{i=1}^{3}((1-0.5) \times(1-0.4) \times(1-0.6))^{\frac{1}{3}} \\
&\left.1-\prod_{i=1}^{3}((1-0.5) \times(1-0.2) \times(1-0.8))^{\frac{1}{3}}, 1-\prod_{i=1}^{3}((1-0.5) \times(1-0.4) \times(1-0.8))^{\frac{1}{3}}\right\} \\
&=\{0.4152,0.4571,0.4687,0.5068,0.5691,0.6085\} \\
& \text { Similarly, others can be obtained as follows: }
\end{aligned}
$$

$h_{1}^{(1)}=\{0.3537,0.4056,0.4354,0.4354,0.4808,0.4870,0.5068,0.5282$, $0.5519,0.5519,0.5879,0.6085\}$
$h_{1}^{(3)}=\{0.4407,0.4687,0.5000,0.5783,0.5879,0.6085,0.6316,0.6893\}$
$h_{2}^{(1)}=\{0.5358,0.5691,0.5783,0.6085,0.7076,0.7286,0.7534,0.8290\}$
$h_{2}^{(2)}=\{0.5358,0.5691,0.6316,0.6580,0.7286,0.7846\}$
$h_{2}^{(3)}=\{0.4407,0.5000,0.5358,0.5783,0.5879,0.6316,0.6580,0.6729$, $0.6893,0.7076,0.7286,0.7534\}$
$h_{3}^{(1)}=\{0.2886,0.3196,0.3537,0.3786,0.4056,0.4152,0.4354,0.4354$, $0.4407,0.4687,0.5068,0.5358\}$
$h_{3}^{(2)}=\{0.2886,0.3160,0.3537,0.3786,0.3918,0.4152,0.4354,0.4687,0.5000\}$
$h_{3}^{(3)}=\{0.2886,0.3458,0.3918,0.4152,0.4407,0.5000\}$

Step 2. Utilize the GHFWA operator (let $\lambda=1$ ) to aggregate all $h_{i}^{(k)}$ ( $k=1,2,3$ ) into a collective HFE $h_{i}$ of the system $A_{i}$ over all the other systems.

Step 3. Calculate the scores $s\left(h_{i}\right)(i=1,2,3)$ :

$$
s\left(h_{1}\right)=0.5364, s\left(h_{2}\right)=0.6471, s\left(h_{3}\right)=0.4018
$$

then $h_{2}>h_{1}>h_{3}$, and thus $A_{2} \succ A_{1} \succ A_{3}$.
If we use the GHFG operator ( $\lambda=1$ ) in Step 1 and the GHFWG operator ( $\lambda=1$ ) in Step 2, then we can get the scores $s\left(h_{i}\right)(i=1,2,3)$ :

$$
s\left(h_{1}\right)=0.4750, s\left(h_{2}\right)=0.5905, s\left(h_{3}\right)=0.3526
$$

which indicate that $A_{2} \succ A_{1} \succ A_{3}$.
From this example, we can see that the hesitant fuzzy preference relation can involve more useful information and can express the uncertain and valueless more deeply. We can also find that we obtain the same conclusion by using the GHFWA and GHFWG operators, but there are slightly differences between them, that is, the former concerns the overall evaluation values while the latter pays more attention to the single one.

### 3.2 Hesitant Multiplicative Preference Relations

Multiplicative preference relation is another important tool to express the DMs' preferences over alternatives described as: let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a set of alternatives, then $B=\left(b_{i j}\right)_{n \times n}$ is called a multiplicative preference relation (Saaty 1980) on $A \times A$, whose element $b_{i j}$ estimates the dominance of the alternative $A_{i}$ over $A_{j}$, and is characterized by a ratio scale such as Saaty's ratio scale such that $b_{i j} \in\left[\frac{1}{9}, 9\right]$, and $b_{i j} b_{j i}=1, i, j=1,2, \cdots, n . b_{i i}=1$ indicates indifference between $A_{j}$ and $A_{i} ; b_{i j}>1$ indicates that $A_{i}$ is preferred to $A_{j}$, especially, $b_{i j}=9$ indicates that $A_{i}$ is absolutely preferred to $A_{j} ; b_{i j}<1$ indicates that $A_{j}$ is preferred to $A_{i}$, especially, $b_{i j}=\frac{1}{9}$ indicates that $A_{j}$ is absolutely preferred to $A_{i}$. If a decision organization constructed by a lot of DMs is asked to provide the estimation of the degrees to which $A_{i}$ is preferred to $A_{j}(i \neq j)$ by using the ratio scale (Saaty 1980), and some DMs provide $r_{i j}^{1}$, some provide $r_{i j}^{2}$
and the others provide $r_{i j}^{3}$, where $r_{i j}^{1}, r_{i j}^{1}, r_{i j}^{3} \in\left[\frac{1}{9}, 9\right]$, then the preference information $r_{i j}$ that $A_{i}$ is preferred to $A_{j}$ can be represented by $r_{i j}=\left\{r_{i j}^{1}, r_{i j}^{2}, r_{i j}^{3}\right\}$ which we call the HME (Xia and Xu 2011c). For the alternatives $A_{i}$ and $A_{k}(k \neq i \neq j)$, some DMs in the decision organization may provide $r_{i k}^{1}$, and the others may provide $r_{i k}^{2}$, then the preference information $r_{i k}$ that $A_{i}$ is preferred to $A_{k}$ can be considered as a HFE $r_{i k}=\left\{r_{i k}^{1}, r_{i k}^{2}\right\}$. For the alternatives $A_{k}$ and $A_{l}(k \neq i \neq j \neq l)$, all the DMs in the decision organization may agree the value $r_{k l}^{1}$, then the preference information $r_{i k}$ that $A_{k}$ is preferred to $A_{l}$ can be considered as a HFE $r_{k l}=\left\{r_{k l}^{1}\right\}$. In such cases, we can not consider the decision organization as one or two or three DMs, but just as a whole providing all the possible preference information about alternatives. Furthermore, all the preference values $r_{i j}(i, j=1,2, \cdots, n)$ can construct the hesitant multiplicative preference relation $R=\left(r_{i j}\right)_{n \times n} \subset A \times A$.

Based on the above analysis, we firstly introduce the definition of hesitant multiplicative preference relation as follows:

Definition 3.2 (Xia and Xu 2013). A hesitant multiplicative preference relation $R$ on the set $A$ is presented by a matrix $R=\left(r_{i j}\right)_{n \times n} \subset A \times A$, where $r_{i j}=$ $\left\{r_{i j}^{t} \mid t=1,2, \cdots, l_{r_{i j}}\right\}$ is a HME indicates that all the possible degrees to which $Y_{i}$ is preferred to $Y_{j}$. Moreover, $r_{i j}$ should satisfy:

$$
\begin{equation*}
r_{i j}^{\sigma(t)} r_{j i}^{\sigma\left(l_{r i i}-t+1\right)}=1, r_{i i}=\{1\}, l_{r_{i j}}=l_{r_{j i}}, i, j=1,2, \cdots, n \tag{3.6}
\end{equation*}
$$

For a group decision making problem, let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ be a discrete set of alternatives, $D_{k}\left(k=1,2, \cdots, p_{0}\right)$ the set of decision organizations and $v=\left(v_{1}, v_{2}, \cdots, v_{p_{0}}\right)^{\mathrm{T}}$ the weight vector of $D_{k}\left(k=1,2, \cdots, p_{0}\right)$, where $\sum_{i=1}^{p_{0}} v_{i}=1$ and $v_{k} \in[0,1], k=1,2, \cdots, p_{0}$. The decision organization $D_{k}$ provides all the possible preference values for each pair of alternatives by using Saaty's scale, and constructs the hesitant multiplicative preference relation
$R_{k}=\left(r_{i j}^{(k)}\right)_{n \times n}$. In the following, we introduce a method for group decision making based on the hesitant multiplicative preference relations (Xia and Xu 2013):

Step 1. Utilize the GHMA (or GHMG) operator to aggregate all $r_{i j}^{(k)}(j=1,2, \cdots, n)$ corresponding to the alternative $A_{i}$, and then get the averaged HME $r_{i}^{(k)}$ of the alternative $A_{i}$ over all the other alternatives for the decision organization $D_{k}$ :

$$
\begin{equation*}
r_{i}^{(k)}=\operatorname{GHMA}_{\lambda}\left(r_{i 1}^{(k)}, r_{i 2}^{(k)}, \cdots, r_{i n}^{(k)}\right)=\left(\frac{1}{n} \bigoplus_{j=1}^{n}\left(r_{i j}^{(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}, i=1,2, \cdots, n \tag{3.7}
\end{equation*}
$$

or
where $\lambda$ is a positive real number.
Step 2. Utilize the GHMWA (or the GHMWG) operator to aggregate all $r_{i}^{(k)}\left(k=1,2, \cdots, p_{0}\right)$ into a collective HME $r_{i}$ of the alternative $A_{i}$ over all the other alternatives:

$$
\begin{equation*}
r_{i}=\text { GHMWA }_{\lambda}\left(R_{i}^{(1)}, R_{i}^{(2)}, \cdots, R_{i}^{\left(p_{0}\right)}\right)=\left(\underset{k=1}{p_{0}} v_{k}\left(r_{i}^{(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}, i=1,2, \cdots, n \tag{3.9}
\end{equation*}
$$

or

$$
\begin{equation*}
r_{i}=\operatorname{GHMWA}_{\lambda}\left(r_{i}^{(1)}, r_{i}^{(2)}, \cdots, r_{i}^{\left(p_{0}\right)}\right)=\left({\underset{k}{p_{0}}}_{\bigotimes_{=1}}^{\left.\left.\left(\lambda r_{i}^{(k)}\right)^{v_{k}}\right)^{\frac{1}{\lambda}}, i=1,2, \cdots, n, n\right]}\right. \tag{3.10}
\end{equation*}
$$

where $\lambda$ is a positive real number.
Step 3. Calculate the scores of $r_{i}(i=1,2, \cdots, n)$, and then rank all the alternatives $A_{i}(i=1,2, \cdots, n)$ and select the best one in accordance with the values of $s\left(r_{i}\right)(i=1,2, \cdots, n)$.

In Example 3.1, if we use hesitant multiplicative preference relations to express the DMs' preferences, then the results are listed in Tables 3.4-3.6 (Xia and Xu 2013).

Table 3.4. The hesitant multiplicative preference relation $R_{1}$
$\left.\left.\begin{array}{ccc}\hline & A_{1} & A_{2} \\ A_{1} & \{1\} & \left\{\frac{1}{8}, \frac{1}{5}, \frac{1}{2}\right\}\end{array}\right\}\{3,5,7,9\}\right\}$

Table 3.5. The hesitant multiplicative preference relation $R_{2}$
$\left.\begin{array}{ccc}\hline & A_{1} & A_{2} \\ A_{1} & \{1\} & \left\{\frac{1}{5}, \frac{1}{2}\right\}\end{array}\right]\left\{\begin{array}{c} \\ A_{2}\end{array}\right.$

Table 3.6. The hesitant multiplicative preference relation $R_{3}$

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{1\}$ | $\left\{\frac{1}{7}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}\right\}$ | $\{2,4\}$ |
| $A_{2}$ | $\{3,4,5,7\}$ | $\{1\}$ | $\{3,8,9\}$ |
| $A_{3}$ | $\left\{\frac{1}{4}, \frac{1}{2}\right\}$ | $\left\{\frac{1}{9}, \frac{1}{8}, \frac{1}{3}\right\}$ | $\{1\}$ |

The following steps are given to get the optimal system:
Step 1. Utilize the GHMA operator (without loss of generality, let $\lambda=1$ ) to aggregate all $r_{i j}^{(k)}(j=1,2,3)$ corresponding to the alternative $A_{i}$, and then get the averaged HME $r_{i}^{(k)}$ of the alternative $A_{i}$ over all the other alternatives for the decision organization $D_{k}$, for example:
$r_{1}^{(1)}=\{1.0801,1.1253,1.2894,1.3811,1.4329,1.6207,1.6207,1.6777,1.8231$, $1.8845,1.8845,2.1072\}$

$$
r_{1}^{(2)}=\{1.4329,1.5612,1.6207,1.7589,1.7850,2.0000\}
$$

$$
r_{1}^{(3)}=\{0.8998,0.9310,0.9574,1.0000,1.2525,1.2894,1.3208,1.3713\}
$$

$$
r_{2}^{(1)}=\{2.1072,2.4760,2.9149,2.9149,3.3795,3.4814,3.9324,4.0133,4.6462\}
$$

$$
r_{2}^{(2)}=\{1.6207,2.1072,2.3019,2.6342,2.9149,2.9149,3.5789,3.9324\}
$$

$$
r_{2}^{(3)}=\{2.1748,2.420,2.6342,3.0000,3.1602,3.3089,3.4814,3.6416,3.7622
$$ $3.9324,4.2415,4.4288\}$

$$
\begin{aligned}
r_{3}^{(1)}= & \{0.3516,0.3644,0.3738,0.3867,0.4057,0.4095,0.4190,0.4363,0.4422, \\
& 0.4598,0.4938\}
\end{aligned}
$$

$$
\begin{aligned}
r_{3}^{(2)}= & \{0.3572,0.3700,0.3738,0.3867,0.3998,0.4116,0.4288,0.4422,0.5000 \\
& 0.5183,0.5326\}
\end{aligned}
$$

$$
r_{3}^{(3)}=\{0.3756,0.4057,0.4618,0.4938\}
$$

Step 2. Utilize the GHMWA operator (let $\lambda=1$ ) to aggregate all $r_{i}^{(k)}$ ( $k=1,2,3$ ) into a collective HME $r_{i}$ of the system $A_{i}$ over all the other systems.

Step 3. Calculate the scores $s\left(r_{i}\right)(i=1,2,3)$ :

$$
s\left(r_{1}\right)=1.3601, s\left(r_{2}\right)=3.1198, s\left(r_{3}\right)=0.4501
$$

then $r_{2}>r_{1}>r_{3}$ and thus $A_{2} \succ A_{1} \succ A_{3}$.
If we use the GHMG operator ( $\lambda=1$ ) in Step 1 and the GHMWG operator ( $\lambda=1$ ) in Step 3, then we can get the scores $s\left(r_{i}\right)(i=1,2,3)$ :

$$
s\left(r_{1}\right)=0.7476, s\left(r_{2}\right)=2.2366, s\left(r_{3}\right)=0.3243
$$

indicating that $A_{2} \succ A_{1} \succ A_{3}$.
Form the above example, we can find that the GHMWA and GHMWG operators can obtain the same result, however, by analyzing the expressions of the GHMWA and GHMWG operators, we can easily find that the GHMWA operator is based the usual arithmetic average which pays more attention to the group opinion and the GHMWG operator is based on the geometric mean which mainly focuses on the individual opinion. In addition, the two methods can get the same result in this example, but in the former, the preference information of any two alternatives is expressed by HFE represented by the values between 0 and 1 which is a uniform distribution around 0.5 reflecting that the preference information of any two alternatives is distributed uniformly, while in the latter, the preference information of any two alternatives is represented by HME constructed by the values between $\frac{1}{9}$ and 9 , which is a non-uniform distribution around 1 reflecting that the preference information of any two alternatives is not distributed uniformly. We can choose the suitable method according the DMs' preferences and the actual situations.

### 3.3 Transitivity and Multiplicative Consistency on Hesitant Fuzzy Preference Relation

In the decision making process, the lack of consistency on a preference relation may lead to an unreasonable result. On the other hand, in practical application, a prefect consistent preference relation is too hard for the DMs to obtain due to the different backgrounds, personal habits, the natural of human judgment, or vague knowledge about the preference degree of one alternative over another (Liao et al. 2013), especially if the number of alternatives is too large (Alonso et al. 2009).

The investigation on consistency of the preference relation can generally involve the following two phases: (1) How to judge whether the preference relation considered is perfectly consistent or acceptably consistent or not; (2) How to adjust or repair the inconsistent preference relation until it is with acceptable consistency. As for the first phase, the concept of consistency has been traditionally defined in
terms of transitivity, such as weak transitivity, max-max transitivity, max-min transitivity, restricted max-min transitivity, restricted max-max transitivity, additive transitivity, and multiplicative transitivity (Tanino 1984, 1988; Szmidt and Kacprzyk 2002; Xu 2007b). Based on the above transitivity properties, some methods for measuring the consistency of a preference relation have been developed (Tanino 1984, 1988; Herrera-Viedma et al. 2004; Ma et al. 2006; Alonso et al. 2008; Dong et al. 2008; Xu and Cai 2011). Saaty (1980) derived a consistency ratio in analytic hierarchy process (AHP), also developed the concept of perfect consistency and acceptable consistency, and pointed out that the preference relation is of acceptable consistency if its consistency ratio is less than 0.1. However, the more common situation in practice is the preference relation possessing unacceptable consistency, which may mislead the ranking result. Therefore, we need to repair the consistency of the preference relation. Liao et al. (2013) utilized the multiplicative consistency to propose some methods for adjusting or repairing the inconsistency of hesitant fuzzy preference relations.

In practical application, in order to choose the most desirable and reasonable solution(s) for a decision making problem, a group of DMs (or experts), who may sometimes come from different aspects, may be gathered together to evaluate the alternatives over the attributes for the sake of avoiding the limited knowledge, personal background, private emotion, and so on. Different DMs can have disagreeing preferences and then it is needed to propose some consensus reaching methods (Montero 1994; Cutello and Montero 1994; Tapia et al. 2012; Xia and Xu 2011d). Liao et al. (2013) developed a consensus improving procedure of hesitant fuzzy preference relations in group decision making.

### 3.3.1 Some Properties of Hesitant Fuzzy Preference Relation

With Definition 3.1, we can easily derive the following result:

Theorem 3.1 (Liao et al. 2013). The transpose $H^{c}=\left(h_{i j}^{c}\right)_{n \times n}$ of the hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$ is also a hesitant fuzzy preference relation, where $h_{i j}^{c}=h_{j i}, i, j=1,2, \cdots, n$.

Theorem 3.2 (Liao et al. 2013). Let $H=\left(h_{i j}\right)_{n \times n}$ be a hesitant fuzzy preference relation. If we remove the $i$ th row and the $i$ th column, then the remaining matrix $\dot{H}=\left(h_{i j}\right)_{(n-1) \times(n-1)}$ is also a hesitant fuzzy preference relation.

When the DM evaluates the preference information, he/she may provide inconsistent preference values and thus constructs the inconsistent preference relation due to the complexity of the considered problem or other reasons.

The hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n} \subset X \times X$ should satisfy the following transitivity properties (Liao et al. 2013):
(1) If $h_{i k} \oplus h_{k j} \geq h_{i j}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies the triangle condition.
(2) If $h_{i k} \geq\{0.5\}, \quad h_{k j} \geq\{0.5\}$, then $h_{i j} \geq\{0.5\}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies the weak transitivity property.
(3) If $h_{i j} \geq \min \left\{h_{i k}, h_{k j}\right\}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies max-min transitivity property.
(4) If $h_{i j} \geq \max \left\{h_{i k}, h_{k j}\right\}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies max-max transitivity property.
(5) If $h_{i k} \geq\{0.5\}, \quad h_{k j} \geq\{0.5\}$, then $h_{i j} \geq \min \left\{h_{i k}, h_{k j}\right\}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies the restricted max-min transitivity property.
(6) If $h_{i k} \geq\{0.5\}, \quad h_{k j} \geq\{0.5\}$, then $h_{i j} \geq \max \left\{h_{i k}, h_{k j}\right\}$, for all $i, j, k=1,2, \ldots, n$, then we say $H$ satisfies the restricted max-max transitivity property.

The weak transitivity is the usual and basic property which can be interpreted as follows: If the alternative $A_{i}$ is preferred to $A_{k}$, and $A_{k}$ is preferred to $A_{j}$, then $A_{i}$ should be preferred to $A_{j}$; If the DM who is logic and consistent does not want to draw inconsistent conclusions, he/she should firstly ensure that the preference relation satisfies the weak transitivity. However, the weak transitivity is the minimum requirement condition to make sure that the hesitant fuzzy preference relation is consistent. There are another two conditions named additive transitivity and multiplicative transitivity which are more restrictive than weak transitivity and can imply reciprocity. The additive transitivity can be generalized to accommodate the hesitant fuzzy preference relation in terms of $\left(h_{i k}-\{0.5\}\right) \oplus\left(h_{k j}-\{0.5\}\right)=\left(h_{i j}-\{0.5\}\right)$, for all $i, j, k=1,2, \ldots, n$. The multiplicative transitivity is an important property of the fuzzy preference relation $U=\left(u_{i j}\right)_{n \times n}$, which was firstly introduced by Tanino (1988) and shown as:

$$
\begin{equation*}
\frac{u_{j i}}{u_{i j}} \cdot \frac{u_{k j}}{u_{j k}}=\frac{u_{k i}}{u_{i k}} \tag{3.11}
\end{equation*}
$$

where $u_{i j}$ denotes a ratio of preference intensity for the alternative $A_{i}$ to that for $A_{j}$, in another words, $A_{i}$ is $u_{i j}$ times as good as $A_{j}$, and $u_{i j} \in[0,1]$, for all $i, j=1,2, \ldots, n$. As to the multiplicative transitivity proposed by Tanino (1988), it might bring some difficulties in meaning, especially some literature assumes an absolute scale for fuzzy sets. To handle this issue, some related work has been done by Montero (1994), Cutello and Montero (1994), et al., devoted to rationality measures and the dimensions of preferences. Due to the existence of intermediate states between extreme rationality and extreme irrationality, Montero (1994) proposed a non-absolutely irrational aggregation rules. After that, Cutello and Montero (1994) extended the rationality measures to fuzzy preference relations. Absolute scale and rationality measures are issues to be further studied, which we will focus on in the future.

Even though both additive transitivity and multiplicative transitivity can be used to measure the consistency, the additive consistency may produce the unreasonable results. Thus, in this section, we shall take the multiplicative transitivity to verify the consistency of a hesitant fuzzy preference relation.

The condition of multiplicative transitivity can be rewritten as follows:

$$
\begin{equation*}
u_{i j} u_{j k} u_{k i}=u_{i k} u_{k j} u_{j i} \tag{3.12}
\end{equation*}
$$

and in the case where $\left(u_{i k}, u_{k j}\right) \notin\{(0,1),(1,0)\}$, Eq.(3.12) is equivalent to the following (Chiclana et al. 2009):

$$
\begin{equation*}
u_{i j}=\frac{u_{i k} u_{k j}}{u_{i k} u_{k j}+\left(1-u_{i k}\right)\left(1-u_{k j}\right)} \tag{3.13}
\end{equation*}
$$

and if $\left(u_{i k}, u_{k j}\right) \in\{(0,1),(1,0)\}$, we stipulate $u_{i j}=0$.
Inspired by Eq.(3.13), Liao et al. (2013) defined the concept of multiplicative consistent hesitant fuzzy preference relation:

Definition 3.3 (Liao et al. 2013). Let $H=\left(h_{i j}\right)_{n \times n}$ be a hesitant fuzzy preference relation on a fixed set $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$, then $H=\left(h_{i j}\right)_{n \times n}$ is multiplicative consistent if

$$
h_{i j}^{\sigma(t)}= \begin{cases}0, & \left(h_{i k}, h_{k j}\right) \in\{(\{0\},\{1\}),(\{1\},\{0\})\} \\ \frac{h_{i k}^{\sigma(t)}(x) h_{k j}^{\sigma(t)}(x)}{h_{i k}^{\sigma(t)}(x) h_{k j}^{\sigma(t)}(x)+\left(1-h_{i k}^{\sigma(t)}(x)\right)\left(1-h_{k j}^{\sigma(t)}(x)\right)}, & \text { otherwise }  \tag{3.14}\\ & \text { for all } i \leq k \leq j\end{cases}
$$

where $h_{i k}^{\sigma(t)}(x)$ and $h_{k j}^{\sigma(t)}(x)$ are the $t$ th smallest values in $h_{i k}(x)$ and $h_{k j}(x)$, respectively.

Theorem 3.3 (Liao et al. 2013). Any hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{2 \times 2}$ is multiplicative consistent.

Proof (Liao et al. 2013). Suppose that $h_{12}=\left\{h_{12}^{\sigma(1)}, h_{12}^{\sigma(2)}, \cdots, h_{12}^{\sigma(n)}\right\}$, then,

$$
\begin{align*}
& h_{21}=\left\{1-h_{12}^{\sigma(n)}, 1-h_{12}^{\sigma(n-1)}, \cdots, 1-h_{12}^{\sigma(1)}\right\} . \text { Thus, } \\
& \frac{0.5 h_{12}^{\sigma(1)}(x)}{0.5 h_{12}^{\sigma(1)}(x)+0.5\left(1-h_{12}^{\sigma(1)}(x)\right)}=\frac{0.5 h_{12}^{\sigma(1)}(x)}{0.5}=h_{12}^{\sigma(1)}(x) \tag{3.15}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \frac{0.5 h_{12}^{\sigma(2)}(x)}{0.5 h_{12}^{\sigma(2)}(x)+0.5\left(1-h_{12}^{\sigma(2)}(x)\right)}=\frac{0.5 h_{12}^{\sigma(2)}(x)}{0.5}=h_{12}^{\sigma(2)}(x)  \tag{3.16}\\
& \vdots  \tag{3.17}\\
& \frac{0.5 h_{12}^{\sigma(n)}(x)}{0.5 h_{12}^{\sigma(n)}(x)+0.5\left(1-h_{12}^{\sigma(n)}(x)\right)}=\frac{0.5 h_{12}^{\sigma(n)}(x)}{0.5}=h_{12}^{\sigma(n)}(x)
\end{align*}
$$

which satisfies Eq.(3.14), additionally, when $h_{12}=\{0\}$, Eq.(3.14) also holds. Thus, $H=\left(h_{i j}\right)_{2 \times 2}$ is multiplicative consistent, which completes the proof of Theorem 3.3.

Based on Definition 3.3 and Theorems 3.1 and 3.3, in order not to increase the dimensions of the derived HFEs in the process of calculations, we can give the following definition:

Definition 3.4 (Liao et al. 2013). Let $H=\left(h_{i j}\right)_{n \times n}$ be a hesitant fuzzy preference relation on a fixed set $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$, then we call
$\bar{H}=\left(\bar{h}_{i j}\right)_{n \times n}$ a prefect multiplicative consistent hesitant fuzzy preference relation, where

$$
\bar{h}_{i j}^{\sigma(t)}(x)=\left\{\begin{array}{l}
\frac{1}{j-i-1} \sum_{k=i+1}^{j-1} \frac{h_{i k}^{\sigma(t)}(x) h_{k j}^{\sigma(t)}(x)}{h_{i k}^{\sigma(t)}(x) h_{k j}^{\sigma(t)}(x)+\left(1-h_{i k}^{\sigma(t)}(x)\right)\left(1-h_{k j}^{\sigma(t)}(x)\right)}, \quad i+1<j  \tag{3.18}\\
h_{i j}^{\sigma(t)}, \quad i+1=j \\
\{0.5\}, \quad i=j \\
1-\bar{h}_{j i}^{\sigma(t)}(x), \quad i>j
\end{array}\right.
$$

and $\bar{h}_{i j}^{\sigma(t)}(x), h_{i k}^{\sigma(t)}(x)$ and $h_{k j}^{\sigma(t)}(x)$ are the $t$ th smallest values in $\bar{h}_{i j}(x)$, $h_{i k}(x)$ and $h_{k j}(x)$, respectively, and $t=1,2, \cdots, l, l=\max \left\{l_{h_{i k}}, l_{h_{k j}}\right\}$.

Definition 3.5 (Liao et al. 2013). Let $H=\left(h_{i j}\right)_{n \times n}$ be a hesitant fuzzy preference relation on a fixed set $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$, then we call $H=\left(h_{i j}\right)_{n \times n}$ an acceptable multiplicative consistent hesitant fuzzy preference relation, if

$$
\begin{equation*}
d(H, \bar{H})<\theta_{0} \tag{3.19}
\end{equation*}
$$

where $d(H, \bar{H})$ is the distance measure between the given hesitant fuzzy preference relation $H$ and its corresponding prefect multiplicative consistent hesitant fuzzy preference relation $\bar{H}$ which can be calculated by Eqs.(3.11) and (3.12), and $\theta_{0}$ is the consistency level. Without loss of generality, we usually let $\theta_{0}=0.1$ in practice.

### 3.3.2 Iterative Algorithm for Improving Consistency of Hesitant Fuzzy Preference Relation

In the general case, the hesitant fuzzy preference relation $H$ constructed by the DM in the decision making problem is generally with unacceptable multiplicative
consistency which means $d(H, \bar{H})>\theta_{0}$. Thus we need to adjust the elements in the hesitant fuzzy preference relation in order to improve the consistency. For the sake of choosing the most desirable alternative, below we propose an iterative algorithm to repair the consistency level of the hesitant fuzzy preference relation (Liao et al. 2013):

## (Algorithm 3.1)

Step 1. Suppose that $k$ is the number of iterations, $\delta$ is the step size, $0 \leq \lambda=k \delta \leq 1$ and $\theta_{0}$ is the consistency level. Let $k=1$, and construct the prefect multiplicative consistent hesitant fuzzy preference relation $\bar{H}=\left(\bar{h}_{i j}\right)_{n \times n}$ from $H^{(k)}=\left(h_{i j}^{(k)}\right)_{n \times n}$ by Eq.(3.18).

Step 2. Calculate the deviation $d\left(H^{(k)}, \bar{H}\right)$ between $\bar{H}$ and $H^{(k)}$ by using:

$$
\begin{equation*}
d_{\text {Hamming }}\left(H^{(k)}, \bar{H}\right)=\frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n}\left[\frac{1}{l_{h_{i j}}} \sum_{t=1}^{l_{h i j}}\left|h_{i j}^{(k) \sigma(t)}-\bar{h}_{i j}^{\sigma(t)}\right|\right] \tag{3.20}
\end{equation*}
$$

or

$$
\begin{equation*}
d_{\text {Euclidean }}\left(H^{(k)}, \bar{H}\right)=\left[\frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\frac{1}{l_{h_{i j}}} \sum_{t=1}^{l_{h_{i j}}}\left|h_{i j}^{(k) \sigma(t)}-\bar{h}_{i j}^{\sigma(t)}\right|^{2}\right)\right]^{\frac{1}{2}} \tag{3.21}
\end{equation*}
$$

where $h_{i j}^{(k) \sigma(t)}$ and $\bar{h}_{i j}^{\sigma(t)}$ are the $t$ th smallest values in $h_{i j}^{(k)}$ and $\bar{h}_{i j}$, respectively. If $d\left(H^{(k)}, \bar{H}\right)<\theta_{0}$, then output $H^{(k)}$; Otherwise, go to the next step.

Step 3. Repair the inconsistent multiplicative hesitant fuzzy preference relation $H^{(k)}$ to $\widehat{H}^{(k)}=\left(h_{i j}^{(k)}\right)_{n \times n}$ by using the following equations:

$$
\begin{gather*}
\widehat{h}_{i j}^{(k) \sigma(t)}=\frac{\left(h_{i j}^{(k) \sigma(t)}\right)^{1-\lambda}\left(\bar{h}_{i j}^{\sigma(t)}\right)^{\lambda}}{\left(h_{i j}^{(k) \sigma(t)}\right)^{1-\lambda}\left(\bar{h}_{i j}^{\sigma(t)}\right)^{\lambda}+\left(1-h_{i j}^{(k) \sigma(t)}\right)^{1-\lambda}\left(1-\bar{h}_{i j}^{\sigma(t)}\right)^{\lambda}}, \\
i, j=1,2, \ldots, n \tag{3.22}
\end{gather*}
$$

where $\widehat{h}_{i j}^{(k) \sigma(t)}, h_{i j}^{(k) \sigma(t)}$ and $\bar{h}_{i j}^{\sigma(t)}$ are the $t$ th smallest values in $\widehat{h}_{i j}^{(k)}, h_{i j}^{(k)}$ and $\bar{h}_{i j}$, respectively. Let $H^{(k+1)}=\widehat{H}^{(k)}$ and $k=k+1$, then go to Step 2.

Example 3.2 (Liao et al. 2013). Suppose that a DM provides his/her preference information over a collection of alternatives $A_{i}(i=1,2,3,4)$ in HFEs and thus constructs the following hesitant fuzzy preference relation:

$$
H=\left(\begin{array}{cccc}
\{0.5\} & \{0.1,0.4\} & \{0.1,0.2\} & \{0.4,0.5,0.6\} \\
\{0.6,0.9\} & \{0.5\} & \{0.3,0.8\} & \{0.3,0.6\} \\
\{0.8,0.9\} & \{0.2,0.7\} & \{0.5\} & \{0.2,0.7\} \\
\{0.4,0.5,0.6\} & \{0.4,0.7\} & \{0.3,0.8\} & \{0.5\}
\end{array}\right)
$$

Firstly, let $k=1$ and $H^{(1)}=H$, then we construct the prefect multiplicative hesitant fuzzy preference relation $\bar{H}=\left(\bar{h}_{i j}\right)_{n \times n}$ from $H^{(1)}$ by Eq.(3.18).

We take $\bar{h}_{14}$ as an example, i.e.,
$\bar{h}_{14}^{\sigma(1)}=\frac{1}{2}\left(\frac{h_{12}^{\sigma(1)} h_{24}^{\sigma(1)}}{h_{12}^{\sigma(1)} h_{24}^{\sigma(1)}+\left(1-h_{12}^{\sigma(1)}\right)\left(1-h_{24}^{\sigma(1)}\right)}+\frac{h_{13}^{\sigma(1)} h_{34}^{\sigma(1)}}{h_{13}^{\sigma(1)} h_{34}^{\sigma(1)}+\left(1-h_{13}^{\sigma(1)}\right)\left(1-h_{34}^{\sigma(1)}\right)}\right)$
$=\frac{1}{2}\left(\frac{0.1 \times 0.3}{0.1 \times 0.3+(1-0.1)(1-0.3)}+\frac{0.1 \times 0.2}{0.1 \times 0.2+(1-0.1)(1-0.2)}\right)$
$=0.036$
$\bar{h}_{14}^{\sigma(2)}=\frac{1}{2}\left(\frac{h_{12}^{\sigma(2)} h_{24}^{\sigma(2)}}{h_{12}^{\sigma(2)} h_{24}^{\sigma(2)}+\left(1-h_{12}^{\sigma(2)}\right)\left(1-h_{24}^{\sigma(2)}\right)}+\frac{h_{13}^{\sigma(2)} h_{34}^{\sigma(2)}}{h_{13}^{\sigma(2)} h_{34}^{\sigma(2)}+\left(1-h_{13}^{\sigma(2)}\right)\left(1-h_{34}^{\sigma(2)}\right)}\right)$
$=\frac{1}{2}\left(\frac{0.4 \times 0.6}{0.4 \times 0.6+(1-0.4)(1-0.6)}+\frac{0.2 \times 0.7}{0.2 \times 0.7+(1-0.2)(1-0.7)}\right)$
$=0.434$
Hence, in the similar way, we can obtain

$$
\bar{H}=\left(\begin{array}{cccc}
\{0.5\} & \{0.1,0.4\} & \{0.046,0.727\} & \{0.036,0.434\} \\
\{0.6,0.9\} & \{0.5\} & \{0.3,0.8\} & \{0.097,0.903\} \\
\{0.273,0.954\} & \{0.2,0.7\} & \{0.5\} & \{0.2,0.7\} \\
\{0.566,0.964\} & \{0.097,0.903\} & \{0.3,0.8\} & \{0.5\}
\end{array}\right)
$$

Then, we use Eq.(3.20) to calculate the hesitant normalized Hamming distance between $H^{(1)}$ and $\bar{H}$ :

$$
\begin{aligned}
& d_{\text {Haaming }}\left(H^{(1)}, \bar{H}\right)=\frac{1}{6} \sum_{i=1}^{4} \sum_{j=1}^{4}\left[\frac{1}{l_{x_{i j}}} \sum_{t=1}^{l_{x_{i j}}}\left|h_{H^{(1)}}^{\sigma(t)}\left(x_{i j}\right)-h_{\bar{H}}^{\sigma(t)}\left(x_{i j}\right)\right|\right] \\
& =\frac{1}{6}\left(\frac{1}{2}(|0.1-0.046|+|0.2-0.727|)+\frac{1}{3}(|0.4-0.036|+|0.5-0.434|\right. \\
& +|0.6-0.5|)+\frac{1}{2}(|0.3-0.097|+|0.6-0.903|)+\frac{1}{2}(|0.8-0.273| \\
& +|0.9-0.954|)+\frac{1}{3}(|0.4-0.5| \\
& +|0.5-0.566|+|0.6-0.964|) \\
& \left.+\frac{1}{2}(|0.4-0.097|+|0.7-0.903|)\right)=0.2071
\end{aligned}
$$

Without loss of generality, let $\theta_{0}=0.1$, then $d_{\text {Hamming }}\left(H^{(1)}, \bar{H}\right)=0.2071<\theta_{0}$, which means that $H^{(1)}$ is not a multiplicative consistent hesitant fuzzy preference relation. Therefore, it needs to repair the inconsistent multiplicative hesitant fuzzy preference relation $H^{(1)}$ according to $\overparen{H}^{(1)}$ by Eq.(3.22), we hereby let $\lambda=0.8$, then

$$
\widehat{H}^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.1,0.4\} & \{0.054,0.624\} & \{0.062,0.447,0.52\} \\
\{0.6,0.9\} & \{0.5\} & \{0.3,0.8\} & \{0.124,0.866\} \\
\{0.376,0.946\} & \{0.2,0.7\} & \{0.5\} & \{0.2,0.7\} \\
\{0.48,0.553,0.938\} & \{0.134,0.876\} & \{0.3,0.8\} & \{0.5\}
\end{array}\right)
$$

We let $H^{(2)}=\widehat{H}^{(1)}$ and $p=2$, then the hesitant normalized Hamming distance between $H^{(2)}$ and $\bar{H}$ can be calculated, i.e., $d_{\text {Hamming }}\left(H^{(2)}, \bar{H}\right)=0.039$ $<0.1$. Since the hesitant normalized Hamming distance is less than the consistency level, we can draw a conclusion that $H^{(2)}$ is the repaired multiplicative consistent hesitant fuzzy preference relation of $H$.

In this example, we can also use Eq.(3.21) to calculate the hesitant normalized Euclidean distance instead of the hesitant normalized Hamming distance, and both of them can get the same result.

Beside Algorithm 3.1, the most directive method for repairing the inconsistency is returning the inconsistent multiplicative hesitant preference relation to the DM to reconsider constructing a new hesitant preference relation according to his/her new comparison until it has acceptable consistency. This algorithm can be described in details as follows (Liao et al. 2013):

## (Algorithm 3.2)

Step 1. See Algorithm 3.1.
Step 2. See Algorithm 3.1.
Step 3. Return the inconsistent multiplicative hesitant fuzzy preference relation $H^{(k)}$ to the DM to reconsider constructing a new hesitant fuzzy preference relation $H^{(k+1)}$ according to the new judgments. Let $k=k+1$, then go to Step 2.

Both of the above two algorithms can guarantee that any multiplicative inconsistent hesitant fuzzy preference relation can be transformed into a hesitant preference relation with acceptable consistency level. But in practice, we usually use the former procedure because the latter may waste a lot of time and resources.

### 3.3.3 Approach to Group Decision Making Based on Multiplicative Consensus of Hesitant Fuzzy Preference Relations

Consider that there exist some same values in a HFE according to Definition 1.1, based on the operational laws in Definition 1.7, the following theorem holds:

Theorem 3.4 (Liao et al. 2013). Suppose that $h_{1}$ and $h_{2}$ are two HFEs, then

$$
\begin{equation*}
l_{h_{1} \oplus h_{2}}=l_{h_{1}} l_{h_{2}}, \quad l_{h_{1} \otimes h_{2}}=l_{h_{1}} l_{h_{2}} \tag{3.23}
\end{equation*}
$$

Similarly, it also holds when there are $n$ different HFEs, i.e.,

$$
\begin{equation*}
\underset{\substack{\underset{i}{i=1} \\ l_{i}}}{ }=\prod_{i=1}^{n} l_{h_{i}}, l_{\substack{\otimes \\ i=1}}^{l_{n}}=\prod_{i=1}^{n} l_{h_{i}} \tag{3.24}
\end{equation*}
$$

From Theorem 3.4, we can see that the dimension of the derived HFE may increase as the addition or multiplicative operations are done, which may increase the complexity of the calculation. Thus, we need to develop some new methods to decrease the dimension of the derived HFE when we operate the HFEs.

Taking Eq.(1.32) as an example, we can obtain $l_{H F W A\left(h_{1}, h_{2}, \cdots, h_{n}\right)}=\prod_{i=1}^{n} l_{h_{i}}$, which is also the same as Eqs.(1.33)-(1.35). Therefore, in order not to increase the dimensions of the derived HFEs in the process of calculations, we firstly adjust the operational laws in Definition 1.7 into the following forms:

For convenience, when two or more HFEs are aggregated, we assume that they have the same length $l$; Otherwise, the short ones should be extended according to the rules as mentioned above.

Definition 3.6 (Liao et al. 2013). Let $h_{j}(j=1,2, \cdots, n)$ be a collection of HFEs, and $\lambda$ a positive real number, then
(1) $h^{\lambda}=\left\{\left(h^{\sigma(t)}\right)^{\lambda}, t=1,2, \cdots, l\right\}$.
(2) $\lambda h=\left\{1-\left(1-h^{\sigma(t)}\right)^{\lambda}, t=1,2, \cdots, l\right\}$.
(3) $h_{1} \oplus h_{2}=\left\{h_{1}^{\sigma(t)}+h_{2}^{\sigma(t)}-h_{1}^{\sigma(t)} h_{2}^{\sigma(t)}, t=1,2, \cdots, l\right\}$.
(4) $h_{1} \otimes h_{2}=\left\{h_{1}^{\sigma(t)} h_{2}^{\sigma(t)}, t=1,2, \cdots, l\right\}$.
(5) $\underset{j=1}{n} h_{j}=\left\{1-\prod_{j=1}^{n}\left(1-h_{j}^{\sigma(t)}\right), t=1,2, \cdots, l\right\}$.
(6) $\bigotimes_{j=1}^{n} h_{j}=\left\{\prod_{j=1}^{n} h_{j}^{\sigma(t)}, t=1,2, \cdots, l\right\}$,
where $h_{j}^{\sigma(t)}$ is the $t$ th smallest value in $h_{j}$.
According to Definition 3.6, we can adjust the aggregation operators (1.32)-(1.35) into the following forms:

Definition 3.7 (Liao et al. 2013). Let $h_{j}(j=1,2, \cdots, n)$ be a collection of HFEs. An adjusted hesitant fuzzy weighted averaging (AHFWA) operator is a mapping $H^{n} \rightarrow H$ such that
$\operatorname{AHFWA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigoplus_{j=1}^{n}\left(w_{j} h_{j}\right)=\left\{1-\prod_{j=1}^{n}\left(1-h_{j}^{\sigma(t)}\right)^{w_{j}} \mid t=1,2, \cdots, l\right\}$
where $h_{j}^{\sigma(t)}$ is the $t$ th smallest value in $h_{j}$, and $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the weight vector of $h_{j}(j=1,2, \cdots, n)$ with $w_{j} \in[0,1], j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$. Especially, if $w=\left(\frac{1}{n}, \frac{1}{n} \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, then the AHFWA operator reduces to the adjusted hesitant fuzzy averaging (AHFA) operator:

$$
\begin{equation*}
\operatorname{AHFA}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigoplus_{j=1}^{n}\left(\frac{1}{n} h_{j}\right)=\left\{\left.1-\prod_{j=1}^{n}\left(1-h_{j}^{\sigma(t)}\right)^{\frac{1}{n}} \right\rvert\, t=1,2, \cdots, l\right\} \tag{3.26}
\end{equation*}
$$

Definition 3.8 (Liao et al. 2013). Let $h_{j}(j=1,2, \cdots, n)$ be a collection of HFEs and let AHFWG: $H^{n} \rightarrow H$, if

$$
\begin{equation*}
\operatorname{AHFWG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigotimes_{j=1}^{n}\left(h_{j}\right)^{w_{j}}=\left\{\prod_{j=1}^{n}\left(h_{j}^{\sigma(t)}\right)^{w_{j}} \mid t=1,2, \cdots, l\right\} \tag{3.27}
\end{equation*}
$$

then AHFWG is called an adjusted hesitant fuzzy weighted geometric (AHFWG) operator, where $h_{j}^{\sigma(t)}$ is the $t$ th smallest value in $h_{j}$, and $w=\left(w_{1}, w_{2} \cdots w_{n}\right)^{\mathrm{T}}$ is the weight vector of $h_{j}(j=1,2, \cdots, n)$, with $w_{j} \in[0,1], \quad j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$. In the case where $w=\left(\frac{1}{n}, \frac{1}{n} \cdots, \frac{1}{n}\right)^{\mathrm{T}}$, the AHFWG operator reduces to the adjusted hesitant fuzzy geometric (AHFG) operator:

$$
\begin{equation*}
\operatorname{AHFG}\left(h_{1}, h_{2}, \cdots, h_{n}\right)=\bigotimes_{j=1}^{n}\left(h_{j}\right)^{\frac{1}{n}}=\left\{\left.\prod_{j=1}^{n}\left(h_{j}^{\sigma(t)}\right)^{\frac{1}{n}} \right\rvert\, t=1,2, \cdots, l\right\} \tag{3.28}
\end{equation*}
$$

In our daily life, in order to choose the most desirable and reasonable solution(s) for a decision making problem, people prefer to form a commitment or organization constructed by several DMs who may sometimes come from different aspects instead of just the single DM for the sake of avoiding the limited knowledge, personal background, private emotion, and so on. As mentioned above, people may provide the preference information by pairwise comparisons and thus construct their preference relations. If people in the commitment or organization express their preference values in HFEs, then some hesitant fuzzy preference relations can be constructed.

The group decision making problem in hesitant fuzzy circumstance can be described as follows:

Suppose that $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ is a discrete set of alternatives; $D_{k}\left(k=1,2, \cdots, p_{0}\right)$ are the decision organizations (each of which contains a collection of DMs), and $v=\left(v_{1}, v_{2}, \cdots, v_{p_{0}}\right)^{\mathrm{T}}$ is the weight vector of the decision organizations with $\sum_{k=1}^{p_{0}} v_{k}=1, v_{k} \in[0,1], k=1,2, \cdots, p_{0}$. The decision organization $D_{k}$ provides all the possible preference values for each pair of alternatives, and constructs a hesitant fuzzy preference relation $H^{(k)}=\left(h_{i j}^{(k)}\right)_{n \times n}\left(k=1,2, \cdots, p_{0}\right)$.

Then, a collective hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$ of $H_{k}=\left(h_{i j}\right)_{n \times n}\left(k=1,2, \cdots, p_{0}\right)$ can be obtained by the AHFWA or AHFWG operator, i.e.,

$$
\begin{array}{r}
h_{i j}=\bigoplus_{k=1}^{p_{0}}\left(w_{k} h_{i j}^{(k)}\right)=\left\{1-\prod_{k=1}^{p_{0}}\left(1-h_{i j}^{(k) \sigma(t)}\right)^{w_{k}} \mid\right. \\
\mid t=1,2, \cdots, l\},  \tag{3.29}\\
i, j=1,2, \cdots, n
\end{array}
$$

and

$$
\begin{equation*}
h_{i j}=\bigotimes_{k=1}^{p_{0}}\left(h_{i j}^{(k)}\right)^{w_{k}}=\left\{\prod_{k=1}^{p_{0}}\left(h_{i j}^{(k) \sigma(t)}\right)^{w_{k}} \mid t=1,2, \cdots, l\right\}, i, j=1,2, \cdots, n \tag{3.30}
\end{equation*}
$$

where $h_{i j}^{(k) \sigma(t)}$ are the $t$ th smallest value in $h_{i j}^{(k)}$.
Now we introduce a consensus improving procedure of hesitant fuzzy preference relations in group decision making (Liao et al. 2013):

## (Algorithm 3.3)

Step 1. Let $\left(H^{(k)}\right)^{(\lambda)}=\left(\left(h_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda)}\left(k=1,2, \cdots, p_{0}\right)$ and $\lambda=1$. We construct the prefect multiplicative consistent hesitant fuzzy preference relations $\left(\bar{H}^{(k)}\right)^{(\lambda)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda)}$ from $\left(H^{(k)}\right)^{(\lambda)}=\left(\left(h_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda)}$ by Algorithm 3.1 (or Algorithm 3.2).

Step 2. Aggregate all the individual prefect multiplicative consistent hesitant fuzzy preference relations $\left(\bar{H}^{(k)}\right)^{(\lambda)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda)}$ into a collective hesitant fuzzy preference relations $(\bar{H})^{(\lambda)}=\left(\left(\bar{h}_{i j}\right)_{n \times n}\right)^{(\lambda)}$ by the AHFWA or AHFWG operator, where
and

$$
\begin{equation*}
\bar{h}_{i j}=\bigotimes_{k=1}^{p_{0}}\left(\bar{h}_{i j}^{(k)}\right)^{w_{k}}=\left\{\prod_{k=1}^{p_{0}}\left(\bar{h}_{i j}^{(k) \sigma(t)}\right)^{w_{k}} \mid t=1,2, \cdots, l\right\}, i, j=1,2, \cdots, n \tag{3.32}
\end{equation*}
$$

where $\bar{h}_{i j}^{(k) \sigma(t)}$ is the $t$ th smallest value in $\bar{h}_{i j}^{(k)}$.
Step 3. Calculate the deviation between each individual hesitant fuzzy preference relation $\left(\bar{H}^{(k)}\right)^{(\lambda)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda)}$ and the collective hesitant fuzzy preference relations $\bar{H}^{(\lambda)}=\left(\left(\bar{h}_{i j}\right)_{n \times n}\right)^{(\lambda)}$, i.e.,

$$
\begin{equation*}
d_{\text {Hamming }}\left(\left(\bar{H}^{(k)}\right)^{(\lambda)}, \bar{H}^{(\lambda)}\right)=\frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n}\left[\frac{1}{l_{\bar{h}_{i j}}} \sum_{t=1}^{\bar{m}_{i j}}\left|\bar{h}_{i j}^{(k)(\lambda) \sigma(t)}-\bar{h}_{i j}^{(\lambda) \sigma(t)}\right|\right] \tag{3.33}
\end{equation*}
$$

or

$$
\begin{equation*}
d_{\text {Euclidean }}\left(\left(\bar{H}^{(k)}\right)^{(\lambda)}, \bar{H}^{(\lambda)}\right)=\left[\frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\frac{1}{l_{h_{i j}}} \sum_{t=1}^{l_{h i j}}\left|h_{i j}^{(k)(\lambda) \sigma(t)}-\bar{h}_{i j}^{(\lambda) \sigma(t)}\right|^{2}\right]^{\frac{1}{2}}\right. \tag{3.34}
\end{equation*}
$$

If $d\left(\left(\bar{H}^{(k)}\right)^{(\lambda)}, \bar{H}^{(\lambda)}\right) \leq \theta_{0}$, for all $k=1,2, \cdots, p_{0}$, where $\theta_{0}$ is the consensus level, then go to Step 5; Otherwise, go to the next step.

Step 4. Let $\left(\bar{H}^{(k)}\right)^{(\lambda+1)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{n \times n}\right)^{(\lambda+1)}$, where

$$
\begin{array}{r}
\bar{h}_{i j}^{(k)(\lambda+1) \sigma(t)}=\frac{\left(\bar{h}_{i j}^{(k)(\lambda) \sigma(t)}\right)^{1-\beta}\left(\bar{h}_{i j}^{(\lambda) \sigma(t)}\right)^{\beta}}{\left(h_{i j}^{(k)(\lambda) \sigma(t)}\right)^{1-\beta}\left(\bar{h}_{i j}^{(\lambda) \sigma(t)}\right)^{\beta}+\left(1-h\left(p_{i j}^{(k)(\lambda) \sigma(t)}\right)^{1-\beta}\left(1-\bar{h}_{i j}^{(\lambda) \sigma(t)}\right)^{\beta}\right.}, \\
i, j=1,2, \ldots, n \tag{3.35}
\end{array}
$$

where $\bar{h}_{i j}^{(k)(\lambda+1) \sigma(t)}, \bar{h}_{i j}^{(k)(\lambda) \sigma(t)}$ and $\bar{h}_{i j}^{(\lambda) \sigma(t)}$ are the $t$ th smallest values in $\bar{h}_{i j}^{(k)(\lambda+1)}, \bar{h}_{i j}^{(k)(\lambda)}$ and $\bar{h}_{i j}^{(\lambda)}$, respectively, $\beta \in(0,1)$. Let $\lambda=\lambda+1$, then go to Step 2.

Step 5. Let $H=\bar{H}^{(\lambda)}$, and employ the AHFA or AHFG operator to fuse all the hesitant preference values $h_{i j}(j=1,2, \ldots, n)$ corresponding to the object $A_{i}$ into the overall hesitant preference value $h_{i}$, i.e.,

$$
\begin{equation*}
h_{i}=\operatorname{AHFA}\left(h_{i 1}, h_{i 2}, \cdots, h_{i n}\right)=\oplus_{j=1}^{n}\left(\frac{1}{n} h_{i j}\right)=\left\{\left.1-\prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{\frac{1}{n}} \right\rvert\, t=1,2, \cdots, l\right\} \tag{3.36}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{i}=\operatorname{AHFG}\left(h_{i 1}, h_{i 2}, \cdots, h_{i n}\right)=\bigotimes_{j=1}^{n}\left(h_{i j}\right)^{\frac{1}{n}}=\left\{\left.\prod_{j=1}^{n}\left(h_{i j}^{\sigma(t)}\right)^{\frac{1}{n}} \right\rvert\, t=1,2, \cdots, l\right\} \tag{3.37}
\end{equation*}
$$

where $h_{i j}^{\sigma(t)}$ is the $t$ th smallest values in $h_{i j}$.
Step 6. Rank all the objects corresponding to the methods given in Section 1.1.

This consensus improving procedure can be interpreted like this: Firstly, we can construct the prefect multiplicative consistent hesitant fuzzy preference relations for individual hesitant fuzzy preference relations given by the different decision organizations; Then a collective hesitant fuzzy preference relation can be obtained by aggregating the constructed prefect multiplicative consistent hesitant fuzzy preference relations. Hence, we can easily calculate the distance between each individual hesitant fuzzy preference relation and the collective hesitant fuzzy preference relation respectively. If the distance is greater than the given consensus level, we need to improve it; Otherwise it is acceptable. To improve the individual hesitant fuzzy preference relation, we fuse it with the collective hesitant fuzzy preference relation by using Eq.(3.36) or (3.37), and then get some new individual hesitant fuzzy preference relations, thus we can iterate until all the individual hesitant fuzzy preference relations are acceptable.

The most directive method for repairing the inconsensus is returning the inconsistent hesitant fuzzy preference relations to the DMs to reconsider constructing new preference relations according to their new comparison until they have acceptable consensus. However, it wastes a lot of time and sometimes this ideal consensus is just a utopian consensus which is difficult to achieve. Based on two soft consensus criteria: the consensus measure and the proximity measure, Tapia García et al. (2012) presented a consensus model for the group decision making problems with interval fuzzy preference relations. They also designed an automatic feedback mechanism to help the DMs in consensus reaching process. Furthermore, Cabrerizo et al. (2009) developed a consensus model for group decision making problems with unbalanced fuzzy linguistic information based on the above two soft consensus criteria. In a multigranular fuzzy linguistic context, Mata et al. (2009) also proposed an adaptive consensus support model for the group decision making problems, which increases the convergence toward the consensus and reduces the number of rounds to reach it. These works are all without interactive. As to our procedure, it is also a decision making aid method with less interaction of the DMs, which can save a lot of time and can give a quick response to the urgent situations.

Indeed, at the beginning of our procedure, we have two improving stages. Firstly, we need to obtain the prefect multiplicative consistent hesitant fuzzy preference relations, which are the improved forms of the hesitant fuzzy preference relations given by the DMs originally. Then we go to iterations, which are also based on improving the preference relations of the DMs. All these steps do not need to reevaluate the alternatives. However, it can make sufficient utilization of the original information.

We now consider a group decision making problem that concerns the evaluation and ranking of the main factors of electronic learning (adapted from Chao and Chen 2009) to illustrate Algorithm 2.3:

Example 3.3 (Liao et al. 2013). As the electronic learning (e-learning) not only can provide expediency for learners to study courses and professional knowledge without the constraint of time and space especially in an asynchronous distance e-learning system, but also may save internal training cost for some enterprises
organizations in a long-term strategy, meanwhile, it also can be used as an alternative self-training for assisting or improving the traditional classroom teaching, the e-learning becomes more and more popular along with the advancement of information technology and has played an important role in teaching and learning not only in different levels of schools but also in various commercial or industrial companies. Many schools and businesses invest manpower and money in e-learning to enhance their hardware facilities and software contents. Thus it is meaningful and urgent to determine which is the most important among the main factors which influence the e-learning effectiveness. Based on the research of Wang (2003) and Tzeng et al. (2007), there are four key factors (or attributes) to evaluate the effectiveness of an e-learning system. These four main factors are: (1) $x_{1}$ : The synchronous learning; (2) $x_{2}$ : The e-learning material; (3) $x_{3}$ : The quality of web learning platform; (4) $x_{4}$ : The self-learning.

In order to rank the above four factors, a committee comprising three DMs $D_{k}(k=1,2,3)$ (whose weight vector is $\left.v=(0.3,0.4,0.3)^{\mathrm{T}}\right)$ is found. After comparing pairs of the factors (or attributes) $x_{i}(i=1,2,3,4)$, the DMs $D_{k}(k=1,2,3)$ give their preferences using HFEs, and then obtain the hesitant fuzzy preference relations as follows:

$$
\begin{aligned}
& H_{1}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.3,0.4\} & \{0.4,0.5,0.6\} & \{0.3,0.7\} \\
\{0.6,0.7,0.8\} & \{0.5\} & \{0.5,0.6\} & \{0.3,0.4\} \\
\{0.4,0.5,0.6\} & \{0.4,0.5\} & \{0.5\} & \{0.4,0.5\} \\
\{0.3,0.7\} & \{0.6,0.7\} & \{0.5,0.6\} & \{0.5\}
\end{array}\right) \\
& H_{2}=\left(\begin{array}{cccc}
\{0.5\} & \{0.3,0.4\} & \{0.5,0.6,0.7\} & \{0.3,0.4,0.6\} \\
\{0.6,0.7\} & \{0.5\} & \{0.4,0.7\} & \{0.4,0.6\} \\
\{0.3,0.4,0.5\} & \{0.3,0.6\} & \{0.5\} & \{0.6,0.7\} \\
\{0.4,0.6,0.7\} & \{0.4,0.6\} & \{0.3,0.4\} & \{0.5\}
\end{array}\right) \\
& H_{3}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.4\} & \{0.4,0.7\} & \{0.3,0.6,0.7\} \\
\{0.6,0.8\} & \{0.5\} & \{0.5,0.7\} & \{0.3,0.6\} \\
\{0.3,0.6\} & \{0.3,0.5,\} & \{0.5\} & \{0.4,0.6\} \\
\{0.3,0.4,0.7\} & \{0.4,0.7\} & \{0.4,0.6\} & \{0.5\}
\end{array}\right)
\end{aligned}
$$

To solve this problem, the following steps are given according to Algorithm 2.3:

Step 1. Let $\left(H^{(k)}\right)^{(\lambda)}=H_{k}$ and $\lambda=1$, we firstly construct respectively the prefect multiplicative consistent hesitant preference relations $\left(\bar{H}^{(k)}\right)^{(1)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{4 \times 4}\right)^{(1)}$ ( $k=1,2,3$ ) from $\left(H^{(k)}\right)^{(1)}=\left(\left(h_{i j}^{(k)}\right)_{4 \times 4}\right)^{(1)} \quad(k=1,2,3)$ by Algorithm 2.1 (or Algorithm 2.2):

$$
\begin{aligned}
& \left(\bar{H}^{(1)}\right)^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.3,0.4\} & \{0.2,0.3,0.5\} & \{0.202,0.361,0.5\} \\
\{0.6,0.7,0.8\} & \{0.5\} & \{0.5,0.6\} & \{0.4,0.6\} \\
\{0.5,0.7,0.8\} & \{0.4,0.5\} & \{0.5\} & \{0.4,0.5\} \\
\{0.5,0.639,0.798\} & \{0.4,0.6\} & \{0.5,0.6\} & \{0.5\}
\end{array}\right) \\
& \left(\bar{H}^{(2)}\right)^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.3,0.4\} & \{0.222,0.609\} & \{0.361,0.596,0.672\} \\
\{0.6,0.7\} & \{0.5\} & \{0.4,0.7\} & \{0.50 .045\} \\
\{0.391,0.778\} & \{0.3,0.6\} & \{0.5\} & \{0.6,0.7\} \\
\{0.328,0.404,0.639\} & \{0.155,0.5\} & \{0.3,0.4\} & \{0.5\}
\end{array}\right) \\
& \left(\bar{H}^{(3)}\right)^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.4\} & \{0.2,0.609\} & \{0.202,0.639\} \\
\{0.6,0.8\} & \{0.5\} & \{0.5,0.7\} & \{0.4,0.778\} \\
\{0.391,0.8\} & \{0.3,0.5\} & \{0.5\} & \{0.4,0.6\} \\
\{0.361,0.798\} & \{0.222,0.6\} & \{0.4,0.6\} & \{0.5\}
\end{array}\right)
\end{aligned}
$$

Step 2. Fuse the individual prefect multiplicative consistent hesitant preference relations $\left(\bar{H}^{(k)}\right)^{(1)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{4 \times 4}\right)^{(1)}$ into a collective prefect hesitant preference relation $(\bar{H})^{(1)}=\left(\left(\bar{h}_{i j}\right)_{4 \times 4}\right)^{(1)}$ by the AHFWA or AHFWG operator. We hereby take the AHFWA operator, i.e., Eq.(3.31), as an example, and then we obtain
$\bar{H}^{(1)}=\left(\begin{array}{cccc}\{0.5\} & \{0.242,0.372,0.472\} & \{0.209,0.447,0.579\} & \{0.27,0.506,0.617\} \\ \{0.532,0.633,0.765\} & \{0.5\} & \{0.462,0.673\} & \{0.442,0.771\} \\ \{0.426,0.571,0.792\} & \{0.332,0.543\} & \{0.5\} & \{0.49,0.619\} \\ \{0.394,0.514,0.745\} & \{0.256,0.563\} & \{0.396,0.53\} & \{0.5\}\end{array}\right)$
Step 3. Calculate the deviation between each individual prefect multiplicative consistent hesitant preference relation $\left(\bar{H}^{(k)}\right)^{(1)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{4 \times 4}\right)^{(1)}$ and the
collective hesitant preference relation $\bar{H}^{(1)}=\left(\left(\bar{h}_{i j}\right)_{4 \times 4}\right)^{(1)}$. In this example, we use Eq.(3.33) i.e. the hesitant normalized Hamming distance, as a representation, and then we have

$$
\begin{gathered}
d_{\text {Hamming }}\left(\left(\bar{H}^{(1)}\right)^{(1)}, \bar{H}^{(1)}\right)=0.162, d_{\text {Hamming }}\left(\left(\bar{H}^{(2)}\right)^{(1)}, \bar{H}^{(1)}\right)=0.128 \\
d_{\text {Hamming }}\left(\left(\bar{H}^{(3)}\right)^{(1)}, \bar{H}^{(1)}\right)=0.07
\end{gathered}
$$

Without loss of generality, we let the consensus level $\theta_{0}=0.1$. We can see that both $d_{\text {Hamming }}\left(\left(\bar{H}^{(1)}\right)^{(1)}, \bar{H}^{(1)}\right)$ and $d_{\text {Hamming }}\left(\left(\bar{H}^{(2)}\right)^{(1)}, \bar{H}^{(1)}\right)$ are bigger than 0.1 , then we need to improve these individual prefect multiplicative consistent hesitant fuzzy preference relations.

Step 4. Let $\beta=0.7$, and by Eq.(3.35), we can construct respectively the new individual hesitant preference relations $\left(\bar{H}^{(k)}\right)^{(2)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{4 \times 4}\right)^{(2)}(k=1,2,3)$ as below:
$\left(\bar{H}^{(1)}\right)^{(2)}=\left(\begin{array}{cccc}\{0.5\} & \{0.229,0.35,0.45\} & \{0.206,0.401,0.556\} & \{0.248,0.461,0.583\} \\ \{0.553,0.654,0.78\} & \{0.5\} & \{0.473,0.652\} & \{0.429,0.725\} \\ \{0.445,0.612,0.794\} & \{0.352,0.53\} & \{0.5\} & \{0.463,0.584\} \\ \{0.425,0.552,0.762\} & \{0.296,0.574\} & \{0.427,0.551\} & \{0.5\}\end{array}\right)$
$\left(\bar{H}^{(2)}\right)^{(2)}=\left(\begin{array}{cccc}\{0.5\} & \{0.259,0.38,0.48\} & \{0.213,0.463,0.588\} & \{0.296,0.533,0.634\} \\ \{0.522,0.623,0.747\} & \{0.5\} & \{0.443,0.681\} & \{0.459,0.796\} \\ \{0.412,0.55,0.788\} & \{0.322,0.56\} & \{0.5\} & \{0.523,0.644\} \\ \{0.374,0.481,0.715\} & \{0.222,0.544\} & \{0.366,0.491\} & \{0.5\}\end{array}\right)$
$\left(\bar{H}^{(3)}\right)^{(2)}=\left(\begin{array}{cccc}\{0.5\} & \{0.229,0.38,0.48\} & \{0.206,0.463,0.588\} & \{0.248,0.504,0.624\} \\ \{0.522,0.623,0.776\} & \{0.5\} & \{0.473,0.681\} & \{0.429,0.773\} \\ \{0.412,0.55,0.794\} & \{0.322,0.53\} & \{0.5\} & \{0.463,0.613\} \\ \{0.384,0.51,0.762\} & \{0.246,0.574\} & \{0.397,0.551\} & \{0.5\}\end{array}\right)$

Let $\lambda=2$, then go back to Step 2 . We fuse the individual hesitant preference relations $\left(\bar{H}^{(k)}\right)^{(2)}(k=1,2,3)$ into a collective hesitant preference relation $(\bar{H})^{(2)}=\left(\left(\bar{h}_{i j}\right)_{4 \times 4}\right)^{(2)}$ by the AHFWA operator (3.31):
$\bar{H}^{(2)}=\left(\begin{array}{cccc}\{0.5\} & \{0.241,0.371,0.471\} & \{0.209,0.445,0.579\} & \{0.268,0.504,0.616\} \\ \{0.532,0.633,0.766\} & \{0.5\} & \{0.461,0.673\} & \{0.442,0.77\} \\ \{0.422,0.57,0.792\} & \{0.331,0.542\} & \{0.5\} & \{0.488,0.618\} \\ \{0.393,0.512,0.744\} & \{0.252,0.562\} & \{0.394,0.528\} & \{0.5\}\end{array}\right)$
Thus, the hesitant normalized Hamming distance between each individual prefect multiplicative consistent hesitant preference relation $\left(\bar{H}^{(k)}\right)^{(2)}$ and the collective hesitant preference relation $\bar{H}^{(2)}$ can be calculated as follows:

$$
\begin{gathered}
d_{\text {Hamming }}\left(\left(\bar{H}^{(1)}\right)^{(2)}, \bar{H}^{(2)}\right)=0.048, d_{\text {Hamming }}\left(\left(\bar{H}^{(2)}\right)^{(2)}, \bar{H}^{(2)}\right)=0.039 \\
d_{\text {Hamming }}\left(\left(\bar{H}^{(3)}\right)^{(2)}, \bar{H}^{(2)}\right)=0.02
\end{gathered}
$$

Now all $d_{\text {Hamming }}\left(\left(\bar{H}^{(k)}\right)^{(2)}, \bar{H}^{(2)}\right)<0.1(k=1,2,3)$, then go to Step 5.

Step 5. Let $H=\bar{H}^{(2)}$, and employ the AHFA or AHFG operator to fuse all the hesitant preference values $h_{i j}(j=1,2, \ldots, n)$ corresponding to the factor $x_{i}$ into the overall hesitant preference value $h_{i}$. We hereby use the AHFA operator to fuse the information. By Eq.(3.36), we have

$$
\begin{aligned}
& h_{1}=\{0.315,0.458,0.545\}, h_{2}=\{0.485,0.537,0.694\} \\
& h_{3}=\{0.444,0.519,0.633\}, h_{4}=\{0.391,0.503,0.597\}
\end{aligned}
$$

Step 6. Using the score function, we can get $s\left(h_{1}\right)=0.439, s\left(h_{2}\right)=0.572$, $s\left(h_{3}\right)=0.532$, and $s\left(h_{4}\right)=0.495$. As $s\left(h_{2}\right)>s\left(h_{3}\right)>s\left(h_{4}\right)>s\left(h_{1}\right)$, we can draw a conclusion that $x_{2} \succ x_{3} \succ x_{4} \succ x_{1}$, which denotes that the e-learning material is the most important factor influencing the affectivity of e-learning.

Surely, in this example, we can also use the AHFWG and AHFG operators to fuse the HFEs in Steps 2 and 5, and we can also use the hesitant normalized Euclidean distance to calculate the deviation between each individual prefect multiplicative consistent hesitant preference relation and the collective prefect multiplicative consistent hesitant fuzzy preference relation in Step 3.

Since our procedure is a decision making aid method which is without interactive, in practical application, the DMs can change their preferences according to our algorithm in order to reach group consensus. For example, in Step 3, as we find that the obtained individual hesitant fuzzy preference relations are not with acceptable consensus, the DMs need to change their preferences. In this case, they can refer to our new constructed individual hesitant fuzzy preference relations $\left(\bar{H}^{(k)}\right)^{(2)}=\left(\left(\bar{h}_{i j}^{(k)}\right)_{4 \times 4}\right)^{(2)} \quad(k=1,2,3)$ and judge whether they are with acceptable consensus or not. Interacting with the DMs frequently during the consensus process is very reliable and accurate but impracticable and time consuming. If the consensus must be urgently obtained, or the DMs cannot or are unwilling to modify their preferences, our procedure is suitable for handling that. Until all hesitant fuzzy preference relations reach group consensus, the selection of the optimal alternative can be derived easily.

Furthermore, from Step 3, we can see that if we take the consensus level within the interval $0.07 \leq \theta_{0} \leq 0.128$, the result will keep the same. In other words, if the consensus level is fixed, small error measurements perhaps do not cause a complete different output, which is to say, our procedures are robust. For example, suppose that the third DM gives his preferences with another hesitant preference relation as:

$$
\dot{H}_{3}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.4\} & \{0.4,0.7\} & \{0.3,0.6,0.7\} \\
\{0.6,0.8\} & \{0.5\} & \{0.6,0.8\} & \{0.3,0.6\} \\
\{0.3,0.6\} & \{0.2,0.4,\} & \{0.5\} & \{0.4,0.6\} \\
\{0.3,0.4,0.7\} & \{0.4,0.7\} & \{0.4,0.6\} & \{0.5\}
\end{array}\right)
$$

In the following, we begin to check whether the output will be changed or not.
Step 1. The prefect multiplicative consistent hesitant preference relations of the third DM can be calculated easily, which is

$$
\left(\dot{\bar{H}}^{(3)}\right)^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.4\} & \{0.2,0.609\} & \{0.202,0.639\} \\
\{0.6,0.8\} & \{0.5\} & \{0.6,0.8\} & \{0.4,0.778\} \\
\{0.391,0.8\} & \{0.2,0.4\} & \{0.5\} & \{0.4,0.6\} \\
\{0.361,0.798\} & \{0.222,0.6\} & \{0.4,0.6\} & \{0.5\}
\end{array}\right)
$$

Step 2. The collective prefect hesitant preference relation can be derived by the AHFWA operator as:
$\dot{\bar{H}}^{(1)}=\left(\begin{array}{cccc}\{0.5\} & \{0.242,0.372,0.472\} & \{0.209,0.447,0.579\} & \{0.27,0.506,0.617\} \\ \{0.532,0.633,0.765\} & \{0.5\} & \{0.497,0.710\} & \{0.442,0.771\} \\ \{0.426,0.571,0.792\} & \{0.332,0.543\} & \{0.5\} & \{0.49,0.619\} \\ \{0.394,0.514,0.745\} & \{0.256,0.563\} & \{0.396,0.53\} & \{0.5\}\end{array}\right)$
Step 3. Calculate the deviation between each individual prefect multiplicative consistent hesitant preference relation and the collective hesitant preference relation $\dot{\bar{H}}^{(1)}$ :

$$
\begin{gathered}
d_{\text {Hamming }}\left(\left(\bar{H}^{(1)}\right)^{(1)}, \dot{\bar{H}}^{(1)}\right)=0.162, d_{\text {Hamming }}\left(\left(\bar{H}^{(2)}\right)^{(1)}, \dot{\bar{H}}^{(1)}\right)=0.130 \\
d_{\text {Hamming }}\left(\left(\dot{\bar{H}}^{(3)}\right)^{(1)}, \dot{\bar{H}}^{(1)}\right)=0.093
\end{gathered}
$$

Since the consensus level $\theta_{0}=0.1$. We can see that both $d_{\text {Hamming }}\left(\left(\bar{H}^{(1)}\right)^{(1)}, \dot{\bar{H}}^{(1)}\right)$ and $d_{\text {Hamming }}\left(\left(\bar{H}^{(2)}\right)^{(1)}, \dot{\bar{H}}^{(1)}\right)$ are bigger than 0.1 , then we need to improve these individual prefect multiplicative consistent hesitant fuzzy preference relations.

Step 4. Let $\beta=0.7$, we can construct respectively the new individual hesitant preference relations $\left(\dot{\bar{H}}^{(k)}\right)^{(2)}(k=1,2,3)$ as:

$$
\begin{aligned}
& \left(\dot{\bar{H}}^{(1)}\right)^{(2)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.229,0.35,0.45\} & \{0.206,0.401,0.556\} & \{0.248,0.461,0.583\} \\
\{0.553,0.654,0.78\} & \{0.5\} & \{0.498,0.679\} & \{0.429,0.725\} \\
\{0.445,0.612,0.794\} & \{0.332,0.512\} & \{0.5\} & \{0.463,0.584\} \\
\{0.425,0.552,0.762\} & \{0.296,0.574\} & \{0.427,0.551\} & \{0.5\}
\end{array}\right) \\
& \left(\dot{\bar{H}}^{(2)}\right)^{(2)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.259,0.38,0.48\} & \{0.213,0.463,0.588\} & \{0.296,0.533,0.634\} \\
\{0.522,0.623,0.747\} & \{0.5\} & \{0.468,0.707\} & \{0.459,0.796\} \\
\{0.412,0.55,0.788\} & \{0.303,0.542\} & \{0.5\} & \{0.523,0.644\} \\
\{0.374,0.481,0.715\} & \{0.222,0.544\} & \{0.366,0.491\} & \{0.5\}
\end{array}\right) \\
& \left(\dot{\bar{H}}^{(3)}\right)^{(2)}=\left(\right)
\end{aligned}
$$

Let $\lambda=2$, then go back to Step 2 . The individual hesitant preference relations $\left(\dot{\bar{H}}^{(k)}\right)^{(2)}(k=1,2,3)$ can be fused into a collective hesitant preference relation $(\dot{\bar{H}})^{(2)}$ by the AHFWA operator:
$\dot{\bar{H}}^{(2)}=\left(\begin{array}{cccc}\{0.5\} & \{0.241,0.371,0.471\} & \{0.209,0.445,0.579\} & \{0.268,0.504,0.616\} \\ \{0.532,0.633,0.766\} & \{0.5\} & \{0.496,0.709\} & \{0.442,0.77\} \\ \{0.422,0.57,0.792\} & \{0.302,0.516\} & \{0.5\} & \{0.488,0.618\} \\ \{0.393,0.512,0.744\} & \{0.252,0.562\} & \{0.394,0.528\} & \{0.5\}\end{array}\right)$
Then, the hesitant normalized Hamming distance between each individual prefect multiplicative consistent hesitant preference relation $\left(\dot{\bar{H}}^{(k)}\right)^{(2)}$ and the collective hesitant preference relation $\dot{\bar{H}}^{(2)}$ can be calculated as:

$$
\begin{gathered}
d_{\text {Hamming }}\left(\left(\dot{\bar{H}}^{(1)}\right)^{(2)}, \dot{\bar{H}}^{(2)}\right)=0.056, d_{\text {Hamming }}\left(\left(\dot{\bar{H}}^{(2)}\right)^{(2)}, \dot{\bar{H}}^{(2)}\right)=0.039 \\
d_{\text {Hamming }}\left(\left(\dot{\bar{H}}^{(3)}\right)^{(2)}, \dot{\bar{H}}^{(2)}\right)=0.023
\end{gathered}
$$

Now all $d_{\text {Hamming }}\left(\left(\dot{\bar{H}}^{(k)}\right)^{(2)}, \dot{\bar{H}}^{(2)}\right)<0.1(k=1,2,3)$, then go to Step 5.

Step 5. Let $\dot{H}=\dot{\bar{H}}^{(2)}$, and employ the AHFA operator to fuse all the hesitant preference values $h_{i j}(j=1,2,3,4)$ corresponding to the factor $x_{i}$ into the overall hesitant preference value $h_{i}$ :

$$
\begin{aligned}
& h_{1}=\{0.315,0.458,0.545\}, h_{2}=\{0.494,0.537,0.703\} \\
& h_{3}=\{0.433,0.519,0.628\}, h_{4}=\{0.391,0.503,0.597\}
\end{aligned}
$$

Step 6. Using the score function, we can get $s\left(h_{1}\right)=0.439, s\left(h_{2}\right)=0.578$, $s\left(h_{3}\right)=0.527$, and $s\left(h_{4}\right)=0.495$. As $s\left(h_{2}\right)>s\left(h_{3}\right)>s\left(h_{4}\right)>s\left(h_{1}\right)$, we can draw a conclusion that $x_{2} \succ x_{3} \succ x_{4} \succ x_{1}$, which denotes that the e-learning material is also the most important factor influencing the affectivity of e-learning.

From the above example, it can be seen that although the third DM slightly changes his/her preference, the output of our procedure also keeps the same. Thus, our algorithm is robust and practicable.

### 3.4 Regression Methods for Hesitant Fuzzy Preference Relations

Since fuzzy preference relations (Orlovsky 1978) have been proven to be an effective tool used in decision making problems (Chiclana et al. 2001; Orlovsky 1978; Tanino 1984), and have close relationship with hesitant fuzzy preference relations, Zhu and Xu (2013a) considered some techniques to transform hesitant fuzzy preference relations into fuzzy preference relations, so as to apply hesitant fuzzy preference relations to decision making through fuzzy preference relations. In order to do so, they concentrated on a selection process of preference degrees in hesitant fuzzy preference relations, which results into the "reduced fuzzy preference relations". Two regression methods were developed to transform hesitant fuzzy preference relations into the reduced fuzzy preference relations based on the additive transitivity and the weak consistency.

The transitivity property is used to represent the idea that the preference degree obtained by directly comparing two alternatives should be equal to or greater than the preference degree between those two alternatives obtained using an indirect chain of alternatives. This property is desirable to avoid contradictions reflected in preference relations. For a fuzzy preference relation $U=\left(u_{i j}\right)_{n \times n}$, Tanino (1984) introduced an additive transitivity as follows:

$$
\begin{equation*}
u_{i j}+u_{j k}=u_{i k}+0.5, \text { for all } i, j, k \tag{3.38}
\end{equation*}
$$

Tanino (1988) also introduced a weak consistency as:

$$
\begin{equation*}
u_{i j} \geq 0.5, u_{j k} \geq 0.5 \rightarrow u_{i k} \geq 0.5, \text { for all } i, j, k \tag{3.39}
\end{equation*}
$$

which means that if the alternative $A_{i}$ is preferred to the alternative $A_{j}$, and the alternative $A_{j}$ is preferred to the alternative $A_{k}$, then the alternative $A_{i}$ should be preferred to the alternative $A_{k}$. This property verifies the condition that a logical and consistent person does not want to express his/her opinions with inconsistency, and is the minimum requirement for consistency.

In the following, we introduce two regression methods for hesitant fuzzy preference relations, which depend on the additive transitivity and the weak consistency respectively.

### 3.4.1 Regression Method of Hesitant Fuzzy Preference Relations Based on Additive Transitivity

Herrera-Viedma et al. (2007) developed a method based on error analysis to measure the consistency levels of fuzzy preference relations. Motivated by this method, Zhu and Xu (2013a) used the additive transitivity and the error analysis to deal with the selection process of hesitant fuzzy preference relations so as to obtain the reduced fuzzy preference relation with the highest consistency level.

Given a hesitant fuzzy preference relation, represented by a matrix $H=\left(h_{i j}\right)_{n \times n}$ $\subset A \times A$, where $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ is a fixed set of alternatives. According to the additive transitivity (3.38), all possible preference degrees of the pair of the alternatives $\left(A_{i}, A_{k}\right)$ represented by a HFE $h_{i k}(i \neq k)$ can be estimated using an intermediate alternative $A_{j}(j \neq i, k)$ as follows:

$$
\begin{equation*}
h_{i k}^{j}=h_{i j} \tilde{+} h_{j k} \simeq 0.5 \tag{3.40}
\end{equation*}
$$

where the operations " $\tilde{+}$ " and " $\simeq$ " are defined as follows:
Definition 3.9 (Zhu and Xu 2013a). Let $h, h_{1}$ and $h_{2}$ be three HFEs, and $b$ a real number, then we have

$$
\begin{gather*}
h_{1} \tilde{+} h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1}+\gamma_{2}\right\}  \tag{3.41}\\
h \simeq b=\bigcup_{\gamma \in h}\{\gamma-b\} \tag{3.42}
\end{gather*}
$$

In order to use Eq.(3.40) to estimate $h_{i k}^{j}$, the alternatives associated with $H$ should generally be classified into several sets defined as follows:

$$
\begin{gather*}
B=\left\{\left(A_{i}, A_{k}\right) \mid i, k \in\{1,2, \ldots, n\} \wedge(i \neq k)\right\} \\
O V^{B}=\left\{\left(A_{i}, A_{k}\right) \in B \mid h_{i k} \text { is an estimated HFE }\right\} \\
K V^{B}=\left(O V^{B}\right)^{c}  \tag{3.45}\\
M_{i k}^{j}=\left\{A_{j} \mid j \neq i, k,\left(A_{i}, A_{j}\right),\left(A_{j}, A_{k}\right) \in K V^{B}\right\} \tag{3.46}
\end{gather*}
$$

where $B$ is a set of all pairs of alternatives, $O V^{B}$ is a set of pairs of alternatives whose preference degrees are represented by the HFE $h_{i k}$, and we call it an estimated HFE; $K V^{B}$ is the complement set of $O V^{B}$ satisfying $K V^{B} \cup O V^{B}=B ; \quad M_{i k}^{j}$ is the set of the intermediate alternatives $A_{j}(j \neq i, k)$ that can be used to estimate $h_{i k}^{j}$ by Eq.(3.40).

Consequently, by Eq.(3.40), we may get several HFEs, $h_{i k}^{j}(j=1,2, \ldots n$, $j \neq i, k)$ indicating all possible estimated preference degrees of the pair of alternatives $\left(A_{i}, A_{k}\right)$. The regression method is to select the proper preference degrees from the estimated HFE $h_{i k}$ for all $i, k=1,2, \ldots, n, i \neq k$ so as to obtain the reduced fuzzy preference relation. For this purpose, we first calculate an average estimated preference degree defined as follows:

$$
\begin{equation*}
\bar{h}_{i k}=\frac{S_{s}\left(\bigcup_{A_{j} \in M_{i k}^{j}} h_{i k}^{j}\right)}{\sum_{A_{j} \in M_{i k}^{j}} l_{h_{i k}^{j}}} \tag{3.47}
\end{equation*}
$$

where $S_{s}$ is a function that indicates the summation of all elements in a set, $l_{h_{i k}^{j}}$ indicates the numbers of all possible preference degrees in $h_{i k}^{j}$.

Comparing the estimated HFE $h_{i k}$ with its average estimated preference degree $\bar{h}_{i k}$, we define the error between them below:

Definition 3.10 (Zhu and Xu 2013a). A HFE indicating all possible errors between an estimated HFE $h_{i k}$ and its average estimated preference degree $\bar{h}_{i k}$ is defined as:

$$
\begin{equation*}
\varepsilon h_{i k}=\frac{2}{3}\left(\bigcup_{\varepsilon_{i k} \in\left(h_{i k}=\bar{h}_{i k}\right)}\left|\varepsilon_{i k}\right|\right) \tag{3.48}
\end{equation*}
$$

where the coefficient $\frac{2}{3}$ is used to make sure each error in $\left|\varepsilon_{i k}\right|$ belongs to the unit interval $[0,1]$.

To choose a preference degree in $h_{i k}$ associated with the lowest error according to Definition 3.10, we define a final preference degree $h_{i k}^{*}$ satisfying

$$
\begin{equation*}
\min \left(\varepsilon h_{i k}\right)=\frac{2}{3}\left(\bigcup_{\varepsilon_{i k} \in\left(h_{i_{i k}} \approx \bar{h}_{i k}\right)}\left|\varepsilon_{i k}\right|\right) \tag{3.49}
\end{equation*}
$$

Collecting $h_{i k}^{*}$ for all $i, k=1,2, \ldots, n, i \neq k$, we get the reduced fuzzy preference relation $\tilde{H}$ transformed from the hesitant fuzzy preference relation $H$. We now introduce the consistency measure to measure the consistency level of $\tilde{H}$.

Definition 3.11 (Zhu and Xu 2013a). For the reduced fuzzy preference relation $\tilde{H}$, the consistency level associated to the final preference degree $h_{i k}^{*}$ is stated as:

$$
\begin{equation*}
c l_{i k}=1-\min \left(\varepsilon h_{i k}\right) \tag{3.50}
\end{equation*}
$$

The consistency level associated to a particular alternative is defined as follows:
Definition 3.12 (Zhu and Xu 2013a). For the reduced fuzzy preference relation $\tilde{H}$, the consistency level associated to a particular alternative $A_{i}$ is stated as:

$$
\begin{equation*}
c l_{i}=\frac{\sum_{i \neq k, k=1}^{n}\left(c l_{i k}+c l_{k i}\right)}{2(n-1)} \tag{3.51}
\end{equation*}
$$

With respect to all alternatives, we have
Definition 3.13 (Zhu and Xu 2013a). The consistency level of $\tilde{H}$ is stated as:

$$
\begin{equation*}
c l_{\tilde{H}}=\frac{\sum_{i=1}^{n} c l_{i}}{n} \tag{3.52}
\end{equation*}
$$

Clearly, the bigger value of $c l_{\tilde{H}}\left(c l_{\tilde{H}} \in[0,1]\right)$, the higher consistency level of $\tilde{H}$.

Based on the analysis above, and assume that a hesitant fuzzy preference relation, represented by a matrix $H=\left(h_{i j}\right)_{n \times n} \subset A \times A$, where $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$ is a fixed set, the algorithm that transforms a hesitant fuzzy preference relation into the reduced fuzzy preference relation with the highest consistency level is developed as follows (Zhu and Xu 2013a):

## (Algorithm 3.4)

Step 1. Randomly locate a HFE, $h_{i k}(i \neq k)$, as the estimated HFE. According to Eq.(2.40), we calculate $h_{i k}^{j}$ for all $j \neq i, k$.
Step 2. Calculate the average estimated preference degree $h_{i k}^{A}$ by Eq.(3.47), and choose a final preference degree $h_{i k}^{*}$ by Eqs.(3.48) and (3.49).
Step 3. Repeat Steps 1 and 2 until all HFEs have been located as the estimated HFEs, then go to the next Step.
Step 4. Saving $h_{i k}^{*}$ for all $i, k \in 1,2, \ldots n, i \neq k$, we have the reduced fuzzy preference relation $\tilde{H}$.
Step 5. Calculating the consistency level of $\tilde{H}$ according to Eqs.(3.50)-(3.52), we get the consistency level of $\tilde{H}$.
Step 6. End.

Example 3.4 (Zhu and Xu 2013a). Assume a hesitant fuzzy preference relation:

$$
H_{1}=\left(\begin{array}{cccc}
\{0.5\} & \{0.4,0.5\} & \{0.6,0.7\} & \{0.6\} \\
\{0.5,0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\
\{0.3,0.4\} & \{0.2\} & \{0.5\} & \{0.1,0.2\} \\
\{0.4\} & \{0.6\} & \{0.8,0.9\} & \{0.5\}
\end{array}\right)
$$

Step 1. Locate the HFE, $h_{12}$, as the estimated HFE. According to Eq.(3.40), we have

$$
h_{12}^{3}=h_{13} \tilde{+} h_{32} \simeq 0.5=\{0.3,0.4\}, h_{12}^{4}=h_{14} \tilde{+} h_{42} \simeq 0.5=\{0.7\}
$$

Step 2. According to Eq.(3.47), we have

$$
\bar{h}_{12}=\frac{S_{s}\left(\sum_{j \in M_{12}^{j}} h_{12}^{j}\right)}{\sum_{j \in M_{12}^{j}} l_{h_{12}^{\prime}}}=\frac{(0.3+0.4)+0.7}{2+1}=0.467
$$

By Eqs.(3.48) and (3.49), we have

$$
\begin{gathered}
\varepsilon h_{12}=\frac{2}{3}\left(\underset{\varepsilon_{12} \in\left(h_{12} \approx \bar{h}_{12}\right)}{\bigcup}\left|\varepsilon_{12}\right|\right)=\{0.044,0.022\} \\
\min \left(\varepsilon h_{12}\right)=0.022=\frac{2}{3}|0.5-0.467|
\end{gathered}
$$

Thus, $h_{12}^{*}=0.5$.

Step 3. Repeat Steps 1 and 2, we have

$$
\begin{gathered}
h_{13}^{*}=0.7, \min \left(\varepsilon h_{13}\right)=0.100 ; h_{14}^{*}=0.6, \min \left(\varepsilon h_{14}\right)=0.189 \\
h_{21}^{*}=0.5, \min \left(\varepsilon h_{21}\right)=0.022 ; h_{23}^{*}=0.8, \min \left(\varepsilon h_{23}\right)=0.056 \\
h_{24}^{*}=0.4, \min \left(\varepsilon h_{24}\right)=0.100 ; h_{31}^{*}=0.3, \min \left(\varepsilon h_{31}\right)=0.067 \\
h_{32}^{*}=0.2, \min \left(\varepsilon h_{32}\right)=0.056 ; h_{34}^{*}=0.2, \min \left(\varepsilon h_{34}\right)=0.089 \\
h_{41}^{*}=0.4, \min \left(\varepsilon h_{41}\right)=0.189 ; h_{42}^{*}=0.6, \min \left(\varepsilon h_{42}\right)=0.100 \\
h_{43}^{*}=0.8, \min \left(\varepsilon h_{43}\right)=0.089
\end{gathered}
$$

Step 4. Saving $h_{i k}^{*}$ for all $i, k=1,2,3,4, i \neq k$, we can get the reduced fuzzy preference relation $\tilde{H}_{1}$ :

$$
\tilde{H}_{1}=\left(\begin{array}{llll}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5
\end{array}\right)
$$

Step 5. According to Eq.(3.52), we have

$$
\begin{gathered}
c l_{12}=0.978, c l_{21}=0.978, c l_{13}=0.900, c l_{31}=0.933, c l_{14}=0.811 \\
c l_{41}=0.811, c l_{23}=0.944, c l_{32}=0.944, c l_{24}=0.900, c l_{42}=0.900 \\
c l_{34}=0.911, c l_{43}=0.911
\end{gathered}
$$

According to Eq.(3.53), we can get

$$
\begin{aligned}
& c l_{1}=\frac{\left(c l_{12}+c l_{21}\right)+\left(c l_{13}+c l_{31}\right)+\left(c l_{14}+c l_{41}\right)}{6}=0.902 \\
& c l_{2}=\frac{\left(c l_{21}+c l_{12}\right)+\left(c l_{23}+c l_{32}\right)+\left(c l_{24}+c l_{42}\right)}{6}=0.941 \\
& c l_{3}=\frac{\left(c l_{31}+c l_{13}\right)+\left(c l_{32}+c l_{23}\right)+\left(c l_{34}+c l_{43}\right)}{6}=0.924 \\
& c l_{4}=\frac{\left(c l_{41}+c l_{14}\right)+\left(c l_{42}+c l_{24}\right)+\left(c l_{43}+c l_{34}\right)}{6}=0.874
\end{aligned}
$$

Furthermore, by Eq.(3.54), the consistency level of a hesitant fuzzy preference relation $\tilde{H}_{1}$ is:

$$
c l_{\tilde{H}_{1}}=\frac{c l_{1}+c l_{2}+c l_{3}+c l_{4}}{4}=0.910
$$

with the consistency level $91.0 \%$.

Step 6. End.

For a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}$, since each preference degree in $h_{i j}$ is a possible value, as a more straightforward method, a hesitant fuzzy preference relation can be separated into all possible fuzzy preference relations, and the fuzzy preference relation with the highest consistency level can be found out by comparing consistency levels of all possible fuzzy preference
relations. In order to illustrate the computation complexity of this "separation method", we give the following example:

Example 3.5 (Zhu and Xu 2013a). Based on the same hesitant fuzzy preference relation, $H_{1}$, we can generate eight possible fuzzy preference relations from $H_{1}$ denoted as follows:

$$
\begin{aligned}
& U_{1}^{H_{1}}=\left(\begin{array}{llll}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5
\end{array}\right), U_{2}^{H_{1}}=\left(\begin{array}{cccc}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5
\end{array}\right) \\
& U_{3}^{H_{1}}=\left(\begin{array}{llll}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5
\end{array}\right), \quad U_{4}^{H_{1}}=\left(\begin{array}{cccc}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5
\end{array}\right) \\
& U_{5}^{H_{1}}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.6 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5
\end{array}\right), \quad U_{6}^{H_{1}}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.7 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5
\end{array}\right) \\
& U_{7}^{H_{1}}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.6 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5
\end{array}\right), U_{8}^{H_{1}}=\left(\begin{array}{cccc}
0.5 & 0.4 & 0.7 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5
\end{array}\right)
\end{aligned}
$$

According to the consistency measure of fuzzy preference relations introduced by Herrera-Viedma et al. (2007), we can get the consistency levels of $U_{i}^{H_{1}}$ ( $i=1,2, \ldots, 8$ ) denoted by

$$
c l_{U_{1}^{H_{1}}}=89.63 \%, c l_{U_{2}^{H_{1}}}=91.76 \%, c_{U_{3}^{H_{1}}}=90.56 \%, c_{U_{4}^{H_{1}}}=92.69 \%
$$

$$
c l_{U_{5}^{H_{1}}}=88.52 \%, c l_{U_{6}^{H_{1}}}=90.65 \%, c l_{U_{1}^{H_{1}}}=89.44 \%, l_{U_{8}^{H_{1}}}=91.57 \%
$$

Since

$$
c l_{U_{4}^{H_{1}}}>c l_{U_{2}^{H_{1}}}>c l_{U_{8}^{H_{1}}}>c l_{U_{6}^{H_{1}}}>c l_{U_{3}^{H_{1}}}>c l_{U_{1}^{H_{1}}}>c l_{U_{7}^{H_{1}}}>c l_{U_{5}^{H_{1}}}
$$

$U_{4}^{H_{1}}$ is the reduced fuzzy preference relation with the highest consistency level.
Obviously, $U_{4}^{H_{1}}=\tilde{H}_{1}$, the same results can be got from the proposed method and the separation method. Comparing Examples 3.4 and 3.5, in order to obtain the reduced fuzzy preference relation with the highest consistency level, the numbers of operational times needed by our regression method and the separation method are $2 n(n-1)+n+1$ and $q(n(n-1)+n+1) \quad(q$ is the number of all possible fuzzy preference relations separated from a hesitant fuzzy preference relation), respectively. Since $q \geq 2$ (at least two fuzzy preference relations can be obtained separated from a hesitant fuzzy preference relation), we have $q(n(n-1)+n+1)>2 n(n-1)+n+1$. The advantage of the regression method is that it is a convenient method to find out a fuzzy preference relation from a hesitant fuzzy preference relation with the highest consistency level quickly. Utilizing the Matlab software for computation, we find that the proposed method can save much more time and is much more effective than the separation method, and the bigger $q$, the more convenient is the regression method.

### 3.4.2 Regression Method of Hesitant Fuzzy Preference Relations Based on Weak Consistency

For the decision making problems in practical applications, the additive transitivity is not necessary due to the complicated environment and the cognitive diversity of humans. But, the weak consistency is essential because a contradictory hesitant fuzzy preference relation doesn't make sense. Therefore, the weak consistency is reasonable if the requirement for consistency is not strict. On the basis of the weak consistency (3.39), we develop a regression method of hesitant fuzzy preference relations which results in fuzzy preference relations satisfying the weak consistency.

In order to propose this regression method, in what follows, we first give the definition of a hesitant preference relation:

Definition 3.14 (Zhu and Xu 2013a). Assume a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}=\left(\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\gamma_{i j}\right\}\right)_{n \times n}$, its hesitant preference degree is defined as:

$$
p_{i j}=\bigcup_{\gamma_{i j} \in h_{i j}}\left\{\delta_{i j} \left\lvert\, \delta_{i j}=\left\{\begin{array}{l}
1,  \tag{3.53}\\
0.5<\gamma_{i j} \leq 1 \\
0, \\
0 \leq \gamma_{i j}<0.5
\end{array}, i, j=1,2, \ldots, n\right\}\right.\right.
$$

where $\delta_{i j}$ is called a hesitant preference element, then $P=\left(p_{i j}\right)_{n \times n}$ is called a hesitant preference relation.

According to graph theory (Bondy and Murtyc 1976), the relationship included in the hesitant preference relation can be reflected by a directed graph, which can be called a "hesitant fuzzy preference graph". In such a graph, each node stands for an alternative, and each directed edge stands for a preference relation. If $p_{i j}=1$, then there is a directed edge from a node $A_{i}$ to a node $A_{j}$, which represents that an alternative $A_{i}$ is superior to an alternative $A_{j}$.

Example 3.6 (Zhu and Xu 2013a). Consider an alternative set $A=\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\}$, and assume two hesitant fuzzy preference relations:

$$
\begin{gathered}
H_{2}=\left(\begin{array}{cccc}
\{0.5\} & \{0.6\} & \{0.6,0.7\} & \{0.6\} \\
\{0.4\} & \{0.5\} & \{0.8\} & \{0.4\} \\
\{0.3,0.4\} & \{0.2\} & \{0.5\} & \{0.2\} \\
\{0.4\} & \{0.6\} & \{0.8\} & \{0.5\}
\end{array}\right) \\
H_{3}=\left(\begin{array}{cccc}
\{0.5\} & \{0.4,0.6\} & \{0.6\} & \{0.4,0.6\} \\
\{0.4,0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\
\{0.4\} & \{0.2\} & \{0.5\} & \{0.2,0.3\} \\
\{0.4,0.6\} & \{0.6\} & \{0.7,0.8\} & \{0.5\}
\end{array}\right)
\end{gathered}
$$

then according to Definition 3.14, we get

$$
P_{H_{2}}=\left(\begin{array}{cccc}
\{0\} & \{1\} & \{1\} & \{1\} \\
\{0\} & \{0\} & \{1\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} \\
\{0\} & \{1\} & \{1\} & \{0\}
\end{array}\right), P_{H_{3}}=\left(\begin{array}{cccc}
\{0\} & \{1,0\} & \{1\} & \{1,0\} \\
\{0,1\} & \{0\} & \{1\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} \\
\{0,1\} & \{1\} & \{1\} & \{0\}
\end{array}\right)
$$

The hesitant fuzzy preference graphs of $P_{H_{2}}$ and $P_{H_{3}}$ are shown in Graphs 3.1 and 3.2 (Zhu and Xu 2013a), respectively.


Graph 3.1. Hesitant fuzzy preference graph $P_{H_{2}}$


Graph 3.2. Hesitant fuzzy preference graph $P_{H_{3}}$

If there is no circular triad in the hesitant fuzzy preference graph, it means that a circular relation of alternatives does not exist, so the corresponding hesitant fuzzy preference relation satisfies the weak consistency, such as Graph 3.1. However, in Graph 3.2, we can see that the alternatives $A_{1}$ and $A_{4}$ are connected by two
opposite directed edges. In such a case, we can get a circular triad of alternatives as $A_{1} \rightarrow A_{4} \rightarrow A_{2} \rightarrow A_{1}$ in Graph 3.2, thus the corresponding hesitant fuzzy preference relation, $H_{3}$, does not satisfy the weak consistency. Therefore, a circular triad can be used to test the weak consistency of a hesitant fuzzy preference relation.

For a hesitant preference relation, a circular triad can be specified as follows:

Definition 3.15 (Zhu and Xu 2013a). Let $P=\left(p_{i j}\right)_{n \times n}$ be the hesitant preference relation of a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, where $p_{i j}$ is given by Eq.(3.53), then

$$
\begin{align*}
& C_{i j k}= \bigcup_{c_{i j k} \in C_{i j k}}\left\{c_{i j k}\right\}= \\
& \bigcup_{\delta_{i j} \in p_{i j}, \delta_{j k} \in p_{j k}, \delta_{k i} \in p_{k i}}\left\{\delta_{i j}+\delta_{j k}+\delta_{k i}\right\}  \tag{3.54}\\
&(i, j, k \in\{1,2, \ldots, n\}, i \neq j \neq k)
\end{align*}
$$

is called a hesitant circular triad power, and $c_{i j k}$ is called a hesitant circular triad power element.

Theorem 3.5 (Zhu and Xu 2013a). For a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}$, we can get its hesitant preference relation $P=\left(p_{i j}\right)_{n \times n}$, and hesitant circular triad power, $C_{i j k}=\bigcup_{c_{i j k} \in C_{i j k}}\left\{c_{i j k}\right\}$. If and only if at least a $c_{i j k}=3$ exists, then $H$ does not satisfy the weak consistency.

Proof. If at least a $c_{i j k}=3$ exists, then at least there exists one circular triad indicating a relation of alternatives as $A_{i} \succ A_{j} \succ A_{k} \succ A_{i}$. According to the definition of the weak consistency (3.39), $H$ does not satisfy the weak consistency; If $H$ does not satisfy the weak consistency, then at least there exists a circular triad of alternatives as $A_{i} \succ A_{j} \succ A_{k} \succ A_{i}$, according to Definition 3.15, we can obtain at least a $c_{i j k}=3$, which complete the proof.

Jiang and Fan (2008) gave a definition of a reachability matrix used to test the weak consistency of fuzzy preference relations. Motivated by this idea, we now develop another method to test the weak consistency of hesitant fuzzy preference relations. Based on the fuzzy preference relation, we define its $k$ th power as follows:

Definition 3.16 (Zhu and Xu 2013a). Let $P=\left(p_{i j}\right)_{n \times n}$ be the hesitant preference relation of a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, then we call $P^{(k)}=\left(p_{i j}^{(k)}\right)_{n \times n}$ the $k$ th power of $P$, where the $(i, j)$ entry, denoted by $p_{i j}^{(k)}$, is the number of different directed edges of the length $k$ from the node $A_{i}$ to the node $A_{j}$.

Furthermore, we define the hesitant reachability matrix as follows:

Definition 3.17 (Zhu and Xu 2013a). Let $P=\left(p_{i j}\right)_{n \times n}$ be the hesitant preference relation of a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, $P^{(3)}=\left(p_{i j}^{(3)}\right)_{n \times n}$ the third power of $P$, then we call the matrix $P^{(3)}$ the hesitant reachability matrix.

Theorem 3.6 (Zhu and Xu 2013a). For a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}$, if all diagonal elements are zero in its hesitant reachability matrix $P^{(3)}=\left(p_{i j}^{(3)}\right)_{n \times n}$, then $H$ satisfies the weak consistency.

Proof. For the hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, if all diagonal elements are zero in its hesitant reachability matrix $P^{(3)}=\left(p_{i j}^{(3)}\right)_{n \times n}$, i.e., $p_{i i}^{(3)}=0, i=1,2, \ldots, n$, it means that there is no circular triad of alternatives in $H$, then $H$ satisfies the weak consistency, which completes the proof.

According to Definition 3.17, and the two hesitant fuzzy preference relations $H_{2}$ and $H_{3}$ in Example 3.6, we can get two hesitant reachability matrices as follows:

$$
P_{H_{2}}^{(3)}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \quad P_{H_{3}}^{(3)}=\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1
\end{array}\right)
$$

By Theorem 3.6, $H_{2}$ has the weak consistency, but $H_{3}$ does not.
Based on the discussion above, Zhu and Xu (2013a) developed an algorithm to obtain the reduced fuzzy preference relations satisfying the weak consistency transformed from a hesitant fuzzy preference relation:

## (Algorithm 3.5)

Given a hesitant fuzzy preference relation, $H^{(q)}=\left(h_{i j}^{(q)}\right)_{n \times n}\left(q=0 ; H^{(q)}\right.$ is the $q$ th power of $H$ indicating the number of being modified). Assume that $H^{(q)}$ can be modified into a preference relation satisfying the weak consistency.

Step 1. According to Definitions 3.14 and 3.17, we can get its hesitant preference relation $P=\left(p_{i j}\right)_{n \times n}$ and the hesitant reachability matrix $P^{(3)}=\left(p_{i j}^{(3)}\right)_{n \times n}$, respectively.

Step 2. According to Theorem 3.6, if $H^{(q)}$ has the weak consistency, turn to Step 5; If $H^{(q)}$ does not satisfy the weak consistency, then turn to Step 3.

Step 3. Refer to the hesitant preference relation $P=\left(p_{i j}\right)_{n \times n}$, then we locate a pair of hesitant preference degrees $\left(p_{i j}, p_{j i}\right)$, satisfying $p_{i j}=\{0,1\}$ and $p_{j i}=\{1,0\}$.

According to Eq.(3.54), we have $C_{i j k}=\bigcup_{c_{i j k} \in C_{i j k}}\left\{c_{i j k}\right\}$, and find $c_{i j k}=3$. Then, we remove the pair of hesitant preference elements $\left(\delta_{i j}, \delta_{j i}\right)$ satisfying $\delta_{i j}+\delta_{j i}=1$, and remove their corresponding preference degrees in the pair of $\operatorname{HFEs}\left(h_{i j}, h_{j i}\right)$ in $H$.

Step 4. Let $q=q+1$, and construct a modified hesitant fuzzy preference relation as $H^{(q+1)}$, then turn to Step 1.

Step 5. Divide $H^{(q)}$ into all possible reduced fuzzy preference relations.

Step 6. End.

Example 3.7 (Zhu and Xu 2013a). Here we use $H_{3}$ in Example 3.6, and let $H_{3}$ be $H_{3}^{(q)}(q=0)$. Since $H_{3}^{(q)}$ does not satisfy the weak consistency, we can directly come to Step 3 of Algorithm 3.5.

Step 3. Locate $\left(p_{14}, p_{41}\right)$, which satisfies $p_{14}=\{0,1\}$ and $p_{41}=\{1,0\}$. According to Eq.(3.54), we have $C_{142}=\{1,2,3\}$, and $C_{143}=\{0,2\}$, where $c_{142}=3$ exists. Thus, we remove the pair of hesitant preference elements $\delta_{14}=\{0\}$ and $\delta_{41}=\{1\}$, and remove their corresponding preference degrees $\gamma_{14}=0.4$ and $\gamma_{41}=0.6$ in the pair of hesitant preference elements $\left(h_{14}, h_{41}\right)$ in $H$.

Step 4. Let $q=q+1$, then we construct a modified hesitant fuzzy preference relation $H_{3}^{(1)}$ as:

$$
H_{3}^{(1)}=\left(\begin{array}{cccc}
\{0.5\} & \{0.4,0.6\} & \{0.6\} & \{0.4\} \\
\{0.4,0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\
\{0.4\} & \{0.2\} & \{0.5\} & \{0.2,0.3\} \\
\{0.6\} & \{0.6\} & \{0.7,0.8\} & \{0.5\}
\end{array}\right)
$$

Then turn to Step 1.

Step 1. According to Definitions 3.14 and 3.17, and the modified hesitant fuzzy preference relation $H_{3}^{(1)}$, we can get its hesitant preference relation $P_{H_{3}^{(1)}}$ and the hesitant reachability matrix $P_{H_{3}^{(1)}}^{(3)}$ respectively:

$$
P_{H_{3}^{(1)}}=\left(\begin{array}{cccc}
\{0\} & \{1,0\} & \{1\} & \{0\} \\
\{0,1\} & \{0\} & \{1\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} \\
\{1\} & \{1\} & \{1\} & \{0\}
\end{array}\right), P_{H_{3}^{(1)}}^{(3)}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

Step 2. According to Theorem 3.6, $H_{3}^{(1)}$ satisfies the weak consistency, turn to Step 5.

Step 5. Divide $H_{3}^{(1)}$ into the following possible reduced fuzzy preference relations satisfying the weak consistency:

$$
\begin{aligned}
& U_{1}^{H_{3}^{(1)}}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.6 & 0.4 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.2 \\
0.6 & 0.6 & 0.8 & 0.5
\end{array}\right), \quad U_{2}^{H_{3}^{(1)}}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.6 & 0.4 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.3 \\
0.6 & 0.6 & 0.7 & 0.5
\end{array}\right) \\
& U_{3}^{H_{3}^{(1)}}=\left(\begin{array}{llll}
0.5 & 0.6 & 0.6 & 0.4 \\
0.4 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.2 \\
0.6 & 0.6 & 0.8 & 0.5
\end{array}\right), \quad U_{4}^{H_{3}^{(1)}}=\left(\begin{array}{llll}
0.5 & 0.6 & 0.6 & 0.4 \\
0.4 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.3 \\
0.6 & 0.6 & 0.7 & 0.5
\end{array}\right)
\end{aligned}
$$

Step 6. End.
In practical applications, Algorithms 3.4 and 3.5 can be combined to obtain a reduced fuzzy preference relation from a hesitant fuzzy preference relation not only satisfying the weak consistency but also having the highest confidence level. That is, to replace Step 5 in Algorithm 3.5 by Algorithm 2.4.

For example, we replace Step 5 in Example 3.7 by Algorithm 3.4, and then obtain a reduced fuzzy preference relation $\tilde{H}_{2}$ with the highest consistency level $95.56 \%$ as:

$$
\tilde{H}_{2}=\left(\begin{array}{llll}
0.5 & 0.4 & 0.6 & 0.4 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.2 \\
0.6 & 0.6 & 0.8 & 0.5
\end{array}\right)
$$

Similar to Example 3.6, we calculate the consistency levels of $U_{i}^{H_{3}^{(1)}}(i=$ $1,2,3,4$ ) according to the method introduced by Herrera-Viedma et al. (2007). Consequently,

$$
\begin{gathered}
c l_{U_{1}^{H_{3}^{(1)}}}=95.93 \%, c l_{U_{2}^{H_{3}^{(1)}}}=94.63 \%, c_{U_{3}^{H_{3}^{(1)}}}=92.96 \% \\
c l_{U_{4}^{H_{3}^{(1)}}}=90.93 \%
\end{gathered}
$$

Obviously, the same result can be obtained, that is $\tilde{H}_{2}=U_{1}^{H_{3}^{(1)}}$.

### 3.5 Deriving a Ranking from Hesitant Fuzzy Preference Relations under Group Decision Making

In group decision making with hesitant fuzzy preference relations, the decision group can provide all possible preferences presented by HFEs, so HFEs often have different numbers of elements. To develop the consistency measures and the priority methods of hesitant fuzzy preference relations, a normalization process becomes necessary to make HFEs have the same number of elements. Two opposite principles can be considered for the normalization (Zhu et al. 2013b): (1) The $\alpha$-normalization, by removing elements of HFEs; 2) The $\beta$-normalization, by adding elements to HFEs. If we aim to select an optimal preference from all possible ones, the $\alpha$-normalization is reasonable; If we want to consider all possible preferences provided by the decision group, we should use the $\beta$-normalization. Consequently, on the basis of the two different normalization principles, we will some different priority methods for hesitant fuzzy preference relations.

### 3.5.1 Deriving Priorities from Hesitant Fuzzy Preference Relations with $\alpha$-Normalization

Based on the $\alpha$-normalization, Zhu et al. (2013b) developed a modelling method to derive priorities from hesitant fuzzy preference relations in group decision making. In the hesitant fuzzy environments, we assume that a group of DMs are hesitant about several possible values for the preference degrees over paired comparisons of alternatives, so as to construct a hesitant fuzzy preference relation.

Suppose a set of alternatives $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}$, and a constructed hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, where $h_{i j}=\left\{h_{i j}^{t} \mid t=1,2, \cdots, l_{h_{i j}}\right\}$. Since each element in $h_{i j}$ is a possible preference degree of the alternative $A_{i}$ over $A_{j}$, then by Eq.(3.12), the consistent preferences can be obtained by

$$
\begin{equation*}
\frac{w_{i}}{w_{i}+w_{j}}=\gamma_{i j}^{\sigma(1)} \text { or } \ldots \text { or } \gamma_{i j}^{\sigma\left(l_{\left.k_{i j}\right)}\right.} \text {, for all } i, j=1,2, \cdots, n \tag{3.55}
\end{equation*}
$$

where $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ is the priority vector of $H$, with $\sum_{i=1}^{n} w_{i}=1$, and $w_{i}>0, i=1,2, \cdots, n ; \gamma_{i j}^{\sigma(t)}\left(t=1, \ldots, l_{h_{i j}}\right)$ is the $t$ th largest element in $h_{i j}$.

Let $\varphi\left(\gamma_{i j}\right)=\gamma_{i j}^{\sigma(1)}$ or $\ldots$ or $\gamma_{i j}^{\sigma\left(l_{h_{i j}}\right)}$, then by Eq.(3.55), we have

$$
\begin{align*}
\frac{w_{i}}{w_{i}+w_{j}}=\varphi\left(\gamma_{i j}\right) & \Leftrightarrow w_{i}=\left(w_{i}+w_{j}\right)\left(\varphi\left(\gamma_{i j}\right)\right) \\
& \Leftrightarrow\left(1-\varphi\left(\gamma_{i j}\right)\right) w_{i}=\varphi\left(\gamma_{i j}\right) w_{j}, i, j=1,2, \cdots, n \tag{3.56}
\end{align*}
$$

Let $1-\varphi\left(\gamma_{i j}\right)=\left(1-\gamma_{i j}^{\sigma(1)}\right)$ or...or $\left(1-\gamma_{i j}^{\sigma\left(l_{l j}\right)}\right)$, then we have $1-\varphi\left(\gamma_{i j}\right)=\varphi\left(\gamma_{j i}\right)$. Thus, Eq.(3.56) can be rewritten as

$$
\begin{equation*}
\frac{w_{i}}{w_{i}+w_{j}}=\varphi\left(\gamma_{i j}\right) \Leftrightarrow \varphi\left(\gamma_{j i}\right) w_{i}=\varphi\left(\gamma_{i j}\right) w_{j}, \quad i, j=1,2, \cdots, n \tag{3.57}
\end{equation*}
$$

Let $\varepsilon_{i j}=\varphi\left(\gamma_{j i}\right) w_{i}-\varphi\left(\gamma_{i j}\right) w_{j}$, then, in order to obtain consistent preferences as much as possible, we minimize $\varepsilon_{i j}$ for all $i$ and $j$. Thus, the priorities of alternatives derived from $H$ can be obtained by solving the following optimization problem (Zhu et al. 2013b):
(M-3.1)

$$
\begin{align*}
& \min \varepsilon_{i j}=\left|\varphi\left(\gamma_{j i}\right) w_{i}-\varphi\left(\gamma_{i j}\right) w_{j}\right|, i, j=1,2, \cdots, n, i \neq j \\
& \text { s.t. } \sum_{i=1}^{n} w_{i}=1, w_{i}>0, i=1,2, \cdots, n \tag{3.58}
\end{align*}
$$

which is called a hesitant multi-objective programming model.
The solution to the hesitant multi-objective programming model is found by solving the following optimization problem (Zhu et al. 2013b):

## (M-3.2)

$$
\begin{align*}
& \min f=\sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n}\left(s_{i j} d_{i j}^{+}+t_{i j} d_{i j}^{-}\right) \\
& \text {s.t. }\left\{\begin{array}{l}
\varphi\left(\gamma_{j i}\right) w_{i}-\varphi\left(\gamma_{i j}\right) w_{j}-s_{i j} d_{i j}^{+}+t_{i j} d_{i j}^{-}=0, i, j=1,2, \cdots, n, i \neq j, \\
\sum_{i=1}^{n} w_{i}=1, w_{i}>0, i=1,2, \cdots, n, \\
d_{i j}^{+}, d_{i j}^{-} \geq 0, i, j=1,2, \cdots, n, i \neq j .
\end{array}\right. \tag{3.59}
\end{align*}
$$

where $d_{i j}^{+}$and $d_{i j}^{-}$are the positive and negative deviations from the target of the goal $\varepsilon_{i j}$, respectively; $s_{i j}$ and $t_{i j}$ are the weights corresponding to $d_{i j}^{+}$and $d_{i j}^{-}$, respectively. We call Eq.(3.59) a hesitant goal programming model.

Without loss of generality, we consider that all the goal functions $\varepsilon_{i j}(i, j=$ $1,2 \ldots, n)$ are fair, then we can set $s_{i j}=t_{i j}=1(i, j=1,2 \ldots, n)$. Consequently, Eq.(3.59) can be rewritten as follows (Zhu et al. 2013b):
(M-3.3)

$$
\begin{align*}
& \min f=\sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n}\left(d_{i j}^{+}+d_{i j}^{-}\right) \\
& \text {s.t. }\left\{\begin{array}{l}
\varphi\left(\gamma_{j i}\right) w_{i}-\varphi\left(\gamma_{i j}\right) w_{j}-d_{i j}^{+}+d_{i j}^{-}=0, i, j=1,2, \cdots, n, i \neq j \\
\sum_{i=1}^{n} w_{i}=1, w_{i}>0, i=1,2, \cdots, n, \\
d_{i j}^{+}, d_{i j}^{-} \geq 0, i, j=1,2, \cdots, n, i \neq j
\end{array}\right. \tag{3.60}
\end{align*}
$$

Solving this model can be considered as a selection process based on the $\alpha$-normalization, and it is to select the optimal preferences from all possible ones for each paired comparison of alternatives until we obtain the deterministic preferences, which results in a fuzzy preference relation consisting of the deterministic preferences.

We now give a numerical example to illustrate the hesitant goal programming model.

Example 3.8 (Zhu et al. 2013b). Assume that a group of DMs provide hesitant preferences over paired comparisons of three alternatives $A_{i}(i=1,2,3)$, so as to construct a hesitant fuzzy preference relation as follows:

$$
H_{1}=\left(\begin{array}{ccc}
\{0.5\} & \{0.2,0.3\} & \{0.6,0.7\} \\
\{0.8,0.7\} & \{0.5\} & \{0.4\} \\
\{0.4,0.3\} & \{0.6\} & \{0.5\}
\end{array}\right)
$$

According to the model (M-3.3), we can build this optimization problem:

$$
\begin{aligned}
& \min f=\sum_{i=1}^{3} \sum_{j=1, i \neq j}^{3}\left(d_{i j}^{+}+d_{i j}^{-}\right) \\
& \text {s.t. }\left\{\begin{array}{l}
(0.8 \text { or } 0.7) w_{1}-(0.2 \text { or } 0.3) w_{2}-d_{12}^{+}+d_{12}^{-}=0 \\
(0.4 \text { or } 0.3) w_{1}-(0.6 \text { or } 0.7) w_{3}-d_{13}^{+}+d_{13}^{-}=0 \\
0.6 w_{2}-0.4 w_{3}-d_{23}^{+}+d_{23}^{-}=0 \\
\sum_{i=1}^{n} w_{i}=1, w_{i}>0, i=1,2,3 \\
d_{i j}^{+}, d_{i j}^{-} \geq 0, i, j=1,2,3, i \neq j
\end{array}\right.
\end{aligned}
$$

The solution of this model can be obtained by a programmed cyclic structure in the Matlab Optimization Tool. The results are listed as follows:

$$
\begin{gathered}
f=0.2488, w_{1}=0.1463, w_{2}=0.3415, w_{3}=0.5122 \\
d_{12}^{+}=d_{12}^{-}=0, d_{13}^{+}=0, d_{13}^{-}=0.2488, d_{23}^{+}=d_{23}^{-}=0
\end{gathered}
$$

Thus, the priority vector of the alternatives is $w_{H_{1}}=(0.1463,0.3415,0.5122)^{\mathrm{T}}$. Thus, the alternative $A_{3}$ is the best. In addition, according to the resolution process of this model, we also can get all final values of membership degrees of $h_{i j}(i, j=1,2,3, i \neq j)$ which result into a fuzzy preference relation as follows:

$$
H_{1}^{\prime}=\left(\begin{array}{lll}
0.5 & 0.3 & 0.6 \\
0.7 & 0.5 & 0.4 \\
0.4 & 0.6 & 0.5
\end{array}\right)
$$

which is called a "reduced hesitant fuzzy preference relation" (Zhu and Xu 2013a).
Based on the principle of $\alpha$-normalization, hesitant fuzzy preference relations can reduce to fuzzy preference relations by removing some membership degrees. So we utilize the consistency measure of fuzzy preference relations to develop this hesitant goal programming model. It's convenient and effective to be operated in practice. However, according to the $\beta$-normalization, we should add some elements to HFEs.

### 3.5.2 Deriving Priorities from Hesitant Fuzzy Preference Relations with $\beta$-Normalization

The $\beta$-normalization aims to add elements to HFEs so as to make them have the same number of preferences in hesitant fuzzy preference relations. In order to derive priorities from hesitant fuzzy preference relations with the $\beta$ normalization, we first develop some consistency measures of hesitant fuzzy preference relations to ensure their consistency, then use the hesitant aggregation operators to aggregate preferences in hesitant fuzzy preference relations to obtain the priorities of alternatives.

To add some elements to HFEs, we use an optimized parameter $\varsigma(0 \leq \varsigma \leq 1)$ originally introduced by Zhu et al. (2013b):

Definition 3.18 (Zhu et al. 2013b). Assume a HFE, $h=\left\{\gamma^{q} \mid q=1, \ldots, l_{h}\right\}$, let $\gamma^{+}$and $\gamma^{-}$be the maximum and minimum membership degrees in $h$ respectively, and $\varsigma(0 \leq \varsigma \leq 1)$ an optimized parameter, then we call $h=\varsigma \gamma^{+} \oplus(1-\varsigma) \gamma^{-}$an added membership degree.

The max, min and the averaged added membership degrees correspond with $\gamma^{*}$, $\gamma_{*}$, and $\gamma_{a}$ respectively, where $\gamma^{*}=\gamma^{+}, \gamma_{*}=\gamma^{-}$, and $\gamma_{a}=0.5\left(\gamma^{+} \oplus \gamma^{-}\right)$. It's clear that $\gamma^{*}$ and $\gamma_{*}$ correspond with the optimism and pessimism rules respectively to add membership degrees to HFEs introduced by Xu and Xia (2011b). The optimized parameter is provided by the group of DMs, and it is used to reflect the DMs' risk preferences: the optimists anticipate desirable outcomes and may add the maximum membership degree, while the pessimists expect unfavorable outcomes and may add the minimum membership degree.

Using $\zeta$, we can add some elements to a hesitant fuzzy preference relation, and get a normalized hesitant fuzzy preference relation defined as follows:

Definition 3.19 (Zhu et al. 2013b). Assume a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}$, and an optimized parameter $\varsigma(0 \leq \varsigma \leq 1)$, where $\varsigma$ is used to add some elements to $h_{i j}(i<j)$, and $1-\varsigma$ is used to add some elements in $h_{j i}(i<j)$ so as to obtain a hesitant fuzzy preference relation, $H^{N}=\left(h_{i j}^{N}\right)_{n \times n}$ satisfying the following conditions:

$$
\begin{equation*}
l_{h_{j}^{v}}=\max \left\{l_{h_{j}^{v}} \mid i, j=1,2, \ldots, n\right\}, \quad i, j=1,2, \ldots, n, i \neq j . \tag{1}
\end{equation*}
$$

(2) $\left(h_{i j}^{N}\right)^{\sigma(q)}+\left(h_{j i}^{N}\right)^{\sigma(q)}=1, h_{i i}=0.5 ; i, j=1,2, \ldots, n$.
(3) $\left(h_{i j}^{N}\right)^{\sigma(q)} \leq\left(h_{i j}^{N}\right)^{\sigma(q+1)},\left(h_{j i}^{N}\right)^{\sigma(q+1)} \leq\left(h_{j i}^{N}\right)^{\sigma(q)}, i<j$.
where $\left(h_{i j}^{N}\right)^{\sigma(q)}$ is the qth element in $h_{i j}^{N}$. Then we call $H^{N}=\left(h_{i j}^{N}\right)_{n \times n}$ a normalized hesitant fuzzy preference relation with the optimized parameter $\varsigma$, and $h_{i j}^{N}$ is a normalized hesitant fuzzy element.

Example 3.9 (Zhu et al. 2013b). For a given hesitant fuzzy preference relation $H_{2}$, according to Definition 3.19, and let $\varsigma=1$, we can get the normalized hesitant fuzzy preference relation $H_{2}^{N}$ as follows:

$$
\begin{aligned}
H_{2} & =\left(\begin{array}{ccc}
\{0.5\} & \{0.3,0.2\} & \{0.6\} \\
\{0.8,0.7\} & \{0.5\} & \{0.6,0.7,0.8\} \\
\{0.4\} & \{0.2,0.3,0.4\} & \{0.5\}
\end{array}\right) \\
H_{2}^{N} & =\left(\begin{array}{ccc}
\{0.5\} & \{0.2,0.3,0.3\} & \{0.6,0.6,0.6\} \\
\{0.8,0.7,0.7\} & \{0.5\} & \{0.6,0.7,0.8\} \\
\{0.4,0.4,0.4\} & \{0.4,0.3,0.2\} & \{0.5\}
\end{array}\right)
\end{aligned}
$$

Based on the normalized hesitant fuzzy preference relations, we now give the definition of consistent hesitant fuzzy preference relation:

Definition 3.20 (Zhu et al. 2013b). Given a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, and an optimized parameter $\varsigma(0 \leq \varsigma \leq 1)$, we can get its normalized hesitant fuzzy preference relation $H^{N}=\left(h_{i j}^{N}\right)_{n \times n}$, if

$$
\begin{align*}
\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\left(h_{j i}^{N}\right)^{\sigma(q)}=\left(h_{k i}^{N}\right)^{\sigma(q)}\left(h_{j k}^{N}\right)^{\sigma(q)}\left(h_{i j}^{N}\right)^{\sigma(q)} \\
i, j, k=1,2, \ldots, n, i \neq j \neq k \tag{3.61}
\end{align*}
$$

where $\left(h_{i j}^{N}\right)^{\sigma(q)}$ is the $q$ th element in $h_{i j}^{N}$, then $H$ is called a consistent hesitant fuzzy preference relation with $\varsigma$ satisfying complete consistency.

Furthermore, by Eq.(3.61), we have

$$
\begin{array}{r}
\left(h_{i j}^{N}\right)^{\sigma(q)}=\frac{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}}{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}+\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)} \\
i, j, k=1,2, \ldots, n, i \neq j \neq k \tag{3.62}
\end{array}
$$

In fact, according Eq.(3.61), we have

$$
\begin{aligned}
& \left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\left(h_{j i}^{N}\right)^{\sigma(q)}=\left(h_{k i}^{N}\right)^{\sigma(q)}\left(h_{j k}^{N}\right)^{\sigma(q)}\left(h_{i j}^{N}\right)^{\sigma(q)} \\
& \Leftrightarrow\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\left(1-\left(h_{i j}^{N}\right)^{\sigma(q)}\right)=\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)\left(h_{i j}^{N}\right)^{\sigma(q)} \\
& \Leftrightarrow\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}-\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\left(h_{i j}^{N}\right)^{\sigma(q)}=\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)\left(h_{i j}^{N}\right)^{\sigma(q)} \\
& \left.\Leftrightarrow\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}=\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)+\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\right)\left(h_{i j}^{N}\right)^{\sigma(q)} \\
& \Leftrightarrow\left(h_{i j}^{N}\right)^{\sigma(q)}=\frac{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}}{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}+\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)} \\
& \text { for all } i, j, k=1,2, \ldots, n, i \neq j \neq k
\end{aligned}
$$

which show that Eq.(3.62) holds.
Theorem 3.7 (Zhu et al. 2013b). Given a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$, and $\varsigma(0 \leq \varsigma \leq 1)$, we can get its normalized hesitant fuzzy preference relation $H^{N}=\left(h_{i j}^{N}\right)_{n \times n}$, for all $i, j, k=1,2, \ldots, n, i \neq j \neq k$, let

$$
\begin{equation*}
\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)}=\frac{\sqrt[n]{\prod_{k=1}^{n}\left(\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{k=1}^{n}\left(\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{k=1}^{n}\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)}} \tag{3.63}
\end{equation*}
$$

then $\bar{H}^{N}=\left(\bar{h}_{i j}^{N}\right)_{n \times n}$ is a consistent hesitant fuzzy preference relation with $\varsigma$.

Proof. For $i, j, k=1,2, \ldots, n, i \neq j \neq k$, let

$$
\left.\begin{array}{l}
\left(h_{i k}^{N}\right)^{\sigma(q)}=\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{i t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t k}^{N}\right)^{\sigma(q)}\right)}} \\
\left(h_{k j}^{N}\right)^{\sigma(q)} \tag{3.65}
\end{array}=\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{k t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}\right)
$$

$$
\begin{equation*}
M_{i k}=\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{i t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t k}^{N}\right)^{\sigma(q)}\right)} \tag{3.66}
\end{equation*}
$$

$$
\begin{equation*}
N_{k j}=\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{k t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t j}^{N}\right)^{\sigma(q)}\right)} \tag{3.67}
\end{equation*}
$$

then

$$
\frac{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}}{\left(h_{i k}^{N}\right)^{\sigma(q)}\left(h_{k j}^{N}\right)^{\sigma(q)}+\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{k j}^{N}\right)^{\sigma(q)}\right)}
$$

$$
\begin{align*}
& =\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)} / M_{i k} N_{k j}}{\left(\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)} / M_{i k} N_{k j}+\right.} \\
& \left(\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\right)} / M_{i k}\right)\left(\sqrt[n]{\left.\prod_{t=1}^{n}\left(1-\left(h_{k t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right) / N_{k j}\right)}\right) \\
& =\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{t i}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\left(h_{t t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{t t}^{N}\right)^{\sigma(q)}\left(h_{t k}^{N}\right)^{\sigma(q)}\left(h_{t t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{t t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{i k}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t i t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{i j}^{N}\right)^{\sigma(q)}\right)}}} \\
& =\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{i t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}=\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)}} \tag{3.68}
\end{align*}
$$

Moreover,

$$
\begin{align*}
& =\binom{\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{i t}^{N}\right)^{\sigma(q)}\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(1-\left(h_{i t}^{N}\right)^{\sigma(q)}\right)\left(1-\left(h_{t j}^{N}\right)^{\sigma(q)}\right)}}}{+\frac{\sqrt[n]{\prod_{t=1}^{n}\left(\left(1-h_{i t}^{N}\right)^{\rho(q)}\left(1-h_{t j}^{N}\right)^{\rho(q)}\right)}}{\sqrt[n]{\prod_{t=1}^{n}\left(\left(1-h_{i t}^{N}\right)^{\sigma(q)}\left(1-h_{t j}^{N}\right)^{\sigma(q)}\right)}+\sqrt[n]{\prod_{t=1}^{n}\left(\left(h_{t j}^{N}\right)^{\sigma(q)}\left(h_{i t}^{N}\right)^{\sigma(q)}\right)}}} \\
& =1 \tag{3.69}
\end{align*}
$$

So we complete the proof of Theorem 3.7.
Theorem 3.8 (Zhu et al. 2013b). For a consistent $H=\left(h_{i j}\right)_{n \times n}$ with $\varsigma$, let $H^{c}=\left(h_{i j}^{c}\right)_{n \times n}$, then $H^{c}$ is also a consistent hesitant fuzzy preference relation with $1-\varsigma$.

Proof. Since $H=\left(h_{i j}\right)_{n \times n}$ is consistent with $\varsigma$, then it means that

$$
\begin{align*}
& \left(h_{i j}^{N}\right)^{\sigma(q)}=\gamma_{i j}^{+} \varsigma+\gamma_{i j}^{-}(1-\varsigma), \quad i<j  \tag{3.70}\\
& \left(h_{j i}^{N}\right)^{\sigma(q)}=\gamma_{j i}^{+}(1-\varsigma)+\gamma_{j i}^{-} \varsigma, \quad j<i \tag{3.71}
\end{align*}
$$

and $H$ satisfies Eq.(3.61), where $\gamma_{i j}^{+}$and $\gamma_{i j}^{-}$are the maximum and minimum membership degrees in $\gamma_{i j}$ respectively.

According to the basic operations on HFEs (see Subsection 1.1.2) we have

$$
\begin{equation*}
\left(\left(h_{i j}^{N}\right)^{\sigma(q)}\right)^{c}=\left(\gamma_{i j}^{+} \varsigma+\gamma_{i j}^{-}(1-\varsigma)\right)^{c} \text {, for all } i, j=1,2, \ldots n \tag{3.72}
\end{equation*}
$$

since $\gamma_{i j}^{+}=1-\gamma_{j i}^{-}$, and $\gamma_{i j}^{-}=1-\gamma_{j i}^{+}$, we can get

$$
\begin{align*}
\left(\left(h_{i j}^{N}\right)^{\sigma(q)}\right)^{c} & =\left(\left(1-\gamma_{j i}^{-}\right) \varsigma+\left(1-\gamma_{j i}^{+}\right)(1-\varsigma)\right)^{c}=\left(\varsigma-\gamma_{j i}^{-} \varsigma+1-\varsigma-\gamma_{j i}^{+}+\gamma_{j i}^{+} \varsigma\right)^{c} \\
& =1-\varsigma+\gamma_{j i}^{-} \varsigma-1+\varsigma+\gamma_{j i}^{+}-\gamma_{j i}^{+} \varsigma \\
& =\gamma_{j i}^{+}(1-\varsigma)+\gamma_{j i}^{-} \varsigma \tag{3.73}
\end{align*}
$$

Similarly, we also can get

$$
\begin{equation*}
\left(\left(h_{j i}^{N}\right)^{\sigma(q)}\right)^{c}=\gamma_{j i}^{+} \varsigma+\gamma_{j i}^{-}(1-\varsigma) \tag{3.74}
\end{equation*}
$$

Since $\left(\left(h_{i j}^{N}\right)^{\sigma(q)}\right)^{c}=\left(h_{j i}^{N}\right)^{\sigma(q)},\left(\left(h_{j i}^{N}\right)^{\sigma(q)}\right)^{c}=\left(h_{i j}^{N}\right)^{\sigma(q)}$, and $H$ is a consistent HFPR with $\varsigma$, then $H^{C}=\left(h_{i j}^{c}\right)_{n \times n}$ is a consistent hesitant fuzzy preference relation with $1-\varsigma$.

This completes the proof of Theorem 3.8.
Theorem 3.9 (Zhu et al. 2013b). Let $H=\left(h_{i j}\right)_{n \times n}$ be a consistent hesitant fuzzy preference relation with $\varsigma$, if we remove the $i$ th row and the $i$ th column for $i=1,2, \ldots, n$, then the remaining hesitant fuzzy preference relation $H^{\prime}=\left(h_{i j}^{\prime}\right)_{(n-1) \times(n-1)}$ is also a consistent hesitant fuzzy preference relation with $\varsigma$.

Proof. Immediately from the definition of a consistent hesitant fuzzy preference relation.

In practical applications, the group of DMs rarely reaches the complete consistency because of some inherent differences among the DMs. Thus, we now consider the acceptable consistency of hesitant fuzzy preference relations.

To define the acceptable consistency, Saaty (1980) developed a consistency ratio $(C R)$ to measure the consistency level of multiplicative preference relations. Then, Crawford and Williams (1985) proposed a geometric consistency index based on a row geometric mean prioritization method. Aguarón and Moreno-Jiménez (2003) further studied the geometric consistency index. But very few researches have focused on the studies of consistency thresholds for fuzzy preference relations.

Dong et al. (2008) used the distance measures to define a consistency index and built the consistency thresholds to identify whether a linguistic preference relation is of acceptable consistency. Motivated by this idea, Zhu et al. (2013b) defined some distance measures of hesitant fuzzy preference relations to identify the consistency levels of hesitant fuzzy preference relations:

Suppose two normalized hesitant fuzzy elements $h_{1}^{N}$ and $h_{2}^{N}$, satisfying $l_{h_{1}^{N}}=l_{h_{2}^{N}}$. Let $l_{h}=l_{h_{1}^{N}}=l_{h_{2}^{N}}$, then the normalized hesitant distance measures can be presented as follows:
(1) The normalized hesitant normalized Hamming distance:

$$
\begin{equation*}
d_{\text {Hamming }}\left(h_{1}^{N}, h_{2}^{N}\right)=\frac{1}{l_{h}}\left(S_{s}\left(\bigcup_{\left(h_{1}^{N}\right)^{\sigma(q)} \in h_{1}^{N},\left(h_{2}^{N}\right)^{\sigma(q)} \in h_{2}^{N}}\left\{\left[\left(h_{1}^{N}\right)^{\sigma(q)}-\left(h_{2}^{N}\right)^{\sigma(q)} \mid\right\}\right)\right)\right. \tag{3.75}
\end{equation*}
$$

where $S_{s}$ is a function that indicates a summation of all values in a set, $\left(h_{1}^{N}\right)^{\sigma(q)}$ and $\left(h_{2}^{N}\right)^{\sigma(q)}$ are the $q$ th largest element in $h_{1}^{N}$ and $h_{2}^{N}$ respectively.
(2) The normalized hesitant normalized Euclidean distance:

$$
\begin{equation*}
d_{\text {Euclidean }}\left(h_{1}^{N}, h_{2}^{N}\right)=\left(\frac{1}{l_{h}}\left(S_{s}\left(\bigcup_{\left(h_{1}^{N}\right)^{\sigma(q)} \in h_{1}^{N},\left(h_{2}^{N}\right)^{\sigma(q)} \in h_{2}^{N}}\left\{\left(\left(h_{1}^{N}\right)^{\sigma(q)}-\left(h_{2}^{N}\right)^{\sigma(q)}\right)^{2}\right\}\right)\right)\right)^{\frac{1}{2}} \tag{3.76}
\end{equation*}
$$

According to the distance measures above and similar to Eq.(3.20), Zhu et al. (2013b) gave the following definition:

Definition 3.21 (Zhu et al. 2013b). Assume two hesitant fuzzy preference relations $H_{1}$ and $H_{2}$ and an optimized parameter $\varsigma$, we get their normalized hesitant fuzzy preference relations $H_{1}^{N}=\left(h_{i j(1)}^{N}\right)_{n \times n}$ and $H_{2}^{N}=\left(h_{i j(2)}^{N}\right)_{n \times n}$ satisfying $l_{h_{i j(1)}^{N}}=l_{h_{i j(2)}^{N}}(i, j=1, \ldots, n, i \neq j)$. Then

$$
\begin{equation*}
d\left(H_{1}, H_{2}\right)=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(d\left(h_{i j(1)}^{N}, h_{i j(2)}^{N}\right)\right)^{2}} \tag{3.77}
\end{equation*}
$$

is called the distance of hesitant fuzzy preference relations with $\varsigma$, where the function $d$ indicates a normalized hesitant distance measure.

Theorem 3.10 (Zhu et al. 2013b). The distance between two hesitant fuzzy preference relations $H_{1}$ and $H_{2}$ denoted by $d\left(H_{1}, H_{2}\right)$ satisfying the following properties:
(1) $0 \leq d\left(H_{1}, H_{2}\right) \leq 1$.
(2) $d\left(H_{1}, H_{2}\right)=0$ if and only if $H_{1}=H_{2}$.
(3) $d\left(H_{1}, H_{2}\right)=d\left(H_{2}, H_{1}\right)$.

Proof. Immediately from the properties of hesitant distance measures (Xu and Xia 2011a).

Assume a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$ and $\varsigma$, according to Theorem 3.7, we can get its consistent hesitant fuzzy preference relation $\bar{H}=\left(\bar{h}_{i j}\right)_{n \times n}$ with $\varsigma$. To make $H$ approximate $\bar{H}$ as much as possible, we set $d(H, \bar{H})$ as the consistency index (CI) with $\varsigma$ of the hesitant fuzzy preference relation $H$ as follows:

$$
\begin{equation*}
C I(H)=d(H, \bar{H}) \tag{3.78}
\end{equation*}
$$

Property 3.1 (Zhu et al. 2013b). $0 \leq C I(H) \leq 1$.
Clearly, the smaller value $C I(H)$, the more consistent the hesitant fuzzy preference relation $H$. If $C I(H)=0$, then $H$ is a consistent hesitant fuzzy preference relation satisfying complete consistency.

Further, by Eqs.(3.77) and (3.78), we have

$$
\begin{equation*}
C I(H)=d(H, \bar{H})=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(d\left(h_{i j}^{N}, \bar{h}_{i j}^{N}\right)\right)^{2}} \tag{3.79}
\end{equation*}
$$

Let the normalized hesitant normalized Hamming distance be the distance measure, Eq.(3.79) can be rewritten as:

$$
\begin{gather*}
C I(H)=d(H, \bar{H})=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(d\left(h_{i j}^{N}, \bar{h}_{i j}^{N}\right)\right)^{2}} \\
=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left[\frac{1}{l_{h}} S_{s}\left({ }_{\left(h_{i j}^{N}\right)^{\sigma(l)} \in h_{i j},\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)} \in \bar{h}_{i j}}\left\{\left(h_{i j}^{N}\right)^{\sigma(q)}-\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)} \mid\right\}\right)\right)^{2}} \tag{3.80}
\end{gather*}
$$

Let

$$
\begin{equation*}
\varepsilon_{i j}=\frac{1}{l_{H}} S_{s}\left(\bigcup_{\left(h_{i j}^{N}\right)^{\sigma(q)} \in h_{i j},\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)} \in \epsilon_{i j}}\left\{\left|\left(h_{i j}^{N}\right)^{\sigma(q)}-\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)}\right|\right\}\right) \tag{3.81}
\end{equation*}
$$

then Eq.(3.81) can be rewritten as:

$$
\begin{equation*}
C I(H)=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(\varepsilon_{i j}\right)^{2}} \tag{3.82}
\end{equation*}
$$

In group decision making, the more consistent the preferences provided by the decision group, the more meaningful the yield of results. According to Eq.(3.79), the values of $C I$ are affected by $\varsigma$ which reflects the DMs' risk preferences. To obtain the highest consistency level, according to Eq.(3.80), we can build the following model to determine the optimal $\varsigma$ :

$$
\min C I(H)=\min d(H, \bar{H})=\min \left(d(H, \bar{H})=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(d\left(h_{i j}^{N}, \bar{h}_{i j}^{N}\right)\right)^{2}}\right)
$$

s.t. $0 \leq \varsigma \leq 1$.

With the optimal $\varsigma$, we can obtain the unique consistent hesitant fuzzy preference relation $\bar{H}$ and the unique $C I(H)$ with the highest consistency level.

When a group of DMs provide preferences over paired comparisons of alternatives, they often have certain consistency tendency (Dong et al. 2008; Zhu et al. 2013b; De Jong 1984). So the values of $\varepsilon_{i j}$ relatively centralize the domain close to zero. Thus, we consider that $\varepsilon_{i j}(i<j)$ are independently and normally distributed with the mean 0 and the standard deviation $\sigma$.

Theorem 3.11 (Zhu et al. 2013b). $\frac{n(n-1)}{2}\left(\frac{1}{\sigma} \times C I(H)\right)^{2}$ is a chi-square distribution with $\frac{n(n-1)}{2}$ degrees of freedom, namely, $\frac{n(n-1)}{2}\left(\frac{1}{\sigma} \times C I(H)\right)^{2} \sim \chi^{2}\left(\frac{n(n-1)}{2}\right)$,
on the condition that $\varepsilon_{i j}(i<j)$ are independent and normally distributed with the mean 0 and the standard deviation $\sigma$, namely, $\varepsilon_{i j} \sim N\left(0, \sigma^{2}\right)$.

Proof. Since

$$
\begin{equation*}
\frac{n(n-1)}{2}\left(\frac{C I(H)}{\sigma}\right)^{2}=\sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(\frac{\varepsilon_{i j}}{\sigma}\right)^{2} \tag{3.84}
\end{equation*}
$$

and $\frac{\varepsilon_{i j}}{\sigma}$ is independent normally distributed with mean 0 and standard deviation 1, consequently, $\frac{n(n-1)}{2}\left(\frac{1}{\sigma} \times C I(H)\right)^{2} \sim \chi^{2}\left(\frac{n(n-1)}{2}\right)$, which completes the proof.

Assume $\sigma^{2}=\sigma_{0}^{2}$, and $\varepsilon_{i j} \sim N\left(0, \sigma_{0}^{2}\right)$, then the consistency measure is to test hypothesis $J_{0}$ versus hypothesis $J_{1}$, denoted by $J_{0}: \sigma^{2} \leq \sigma_{0}^{2} ; J_{1}$ : $\sigma^{2}<\sigma_{0}^{2}$. The degrees of freedom of the estimator $\chi^{2}=\sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(\frac{\varepsilon_{i j}}{\sigma}\right)^{2}$ is $\frac{n(n-1)}{2}$, this is an one-sided right-tailed test, and we can get the critical value $\lambda_{\alpha_{0}}$ of $\chi^{2}$ distribution at the significance level $\alpha_{0}$. Consequently we have

$$
\begin{equation*}
C \bar{I}(H)=\sigma_{0} \sqrt{\frac{2}{n(n-1)} \lambda_{\alpha_{0}}} \tag{3.85}
\end{equation*}
$$

If $C I(H) \leq C \bar{I}(H)$, then $H$ is a hesitant fuzzy preference relation with the acceptable consistency; If $C I(H)>C \bar{I}(H)$, then the consistency level of $H$ is unacceptable.

The parameters $\alpha_{0}$ and $\sigma_{0}$ are determined by the DMs according to actual situations. Table 3.7 (Zhu et al. 2013b) shows the values of $C \bar{I}(H)$ for different $n$ with $\alpha=0.1$ and $\sigma_{0}=0.2$.

Table 3.7. The values of $C \bar{I}(H)\left(\alpha_{0}=0.1, \sigma_{0}=0.2\right)$

| $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0882 | 0.1211 | 0.1396 | 0.1510 | 0.1586 | 0.1643 |

For the hesitant fuzzy preference relations with unacceptable consistency, we present a modeling method to improve the consistency.

Assume a hesitant fuzzy preference relation $H=\left(h_{i j}\right)_{n \times n}$ with the unacceptable consistency, according to Eq.(3.83), we can obtain its consistent hesitant fuzzy preference relation $\bar{H}=\left(\bar{h}_{i j}\right)_{n \times n}$ and the optimized parameter $\varsigma$. Let ${ }^{m} H^{N}=$ $\left({ }^{m} h_{i j}^{N}\right)_{n \times n}$ be the modified normalized hesitant fuzzy preference relation with the acceptable consistency, where

$$
\begin{equation*}
\left({ }^{m} h_{i j}^{N}\right)^{\sigma(q)}=\left(h_{i j}^{N}\right)^{\sigma(q)}+\left(x_{i j}\right)^{\sigma(q)}, i, j=1,2, \ldots, n, i<j \tag{3.86}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{i j}=\left\{\left(x_{i j}\right)^{\sigma(q)} \mid q=1, \ldots, l_{h_{i j}^{N}}, i<j\right\} \tag{3.87}
\end{equation*}
$$

is the set of adjusted valuables, then we build the model as follows:

$$
\begin{equation*}
\min \left(\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n} S_{s}\left(x_{i j}\right)^{2}\right) \tag{3.88}
\end{equation*}
$$

with the conditions that

$$
\left\{\begin{array}{l}
x_{i j}^{\sigma(q)}+x_{j i}^{\sigma(q)}=0,  \tag{3.89}\\
C I\left({ }^{m} H\right) \leq C \bar{I}(H),
\end{array}\right.
$$

where
$C I\left({ }^{m} H\right)=\sqrt{\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n}\left(\frac{1}{l_{h}} S_{s}\left({ }_{\left(h_{i}^{N}\right)^{(\gamma)} \in h_{i j},\left(\overline{\bar{h}}_{j}^{N}\right)^{\sigma(q)} \in \bar{h}_{i j}}\left\{\left\{\left({ }^{m} h_{i j}^{N}\right)^{\sigma(q)}-\left(\bar{h}_{i j}^{N}\right)^{\sigma(q)} \mid\right\}\right)\right)^{2}\right.}$

Therefore, we can build an optimization model as follows:

$$
\begin{align*}
& \min \left(\frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n} S_{s}\left(x_{i j}\right)^{2}\right) \\
& \text { s.t. }\left\{\begin{array}{l}
x_{i j}^{\sigma(q)}+x_{j i}^{\sigma(q)}=0, \\
C I\left({ }^{m} H\right) \leq C \bar{I}(H) .
\end{array}\right. \tag{3.91}
\end{align*}
$$

Based on the discussion above, we give the following algorithm to obtain a hesitant fuzzy preference relation with the acceptable consistency (Zhu et al. 2013b):

## (Algorithm 3.6)

Step 1. Assume a hesitant fuzzy preference relation, $H=\left(h_{i j}\right)_{n \times n}$, and an optimized parameter $\varsigma(0 \leq \varsigma \leq 1)$, we can obtain the optimal $\varsigma$ and $\min C I(H)$ according to Eq.(3.83). Based on Table 3.7, if $\min C I(H)>C \bar{I}(H)$, then go to Step 2; Otherwise, go to Step 3 .

Step 2. Using Eq.(3.91), we can get the modified normalized hesitant fuzzy preference relation ${ }^{m} H^{N}$ and $C I\left({ }^{m} H\right)$, where $C I\left({ }^{m} H\right) \leq C \bar{I}(H)$.

Step 3. End.

Example 3.10 (Zhu et al. 2013b). Assume a hesitant fuzzy preference relation $H_{3}$ denoted as:

$$
H_{3}=\left(\begin{array}{cccc}
\{0.5\} & \{0.8\} & \{0.3\} & \{0.6,0.7\} \\
\{0.2\} & \{0.5\} & \{0.5\} & \{0.8\} \\
\{0.7\} & \{0.5\} & \{0.5\} & \{0.2,0.3,0.4\} \\
\{0.4,0.3\} & \{0.2\} & \{0.8,0.7,0.6\} & \{0.5\}
\end{array}\right)
$$

Using Eq.(3.83), we have

$$
\varsigma=0, \min C I\left(H_{3}\right)=\min d\left(H_{3}, \bar{H}_{3}\right)=0.1781
$$

and the consistent hesitant fuzzy preference relation $\bar{H}_{3}$ of $H_{3}$. Then according to Table 3.7, we have $\min C I\left(H_{3}\right)>C \bar{I}\left(H_{3}\right)=0.1211$. So $H_{3}$ needs to be optimized.

Using Eq.(3.91), we can get the modified normalized hesitant fuzzy preference relation ${ }^{m} H_{3}^{N}$ :

$$
{ }^{m} H_{3}^{N}=\left(\begin{array}{cccc}
\{0.5\} & \{0.7264,0.7264,0.7342\} & \{0.3867,0.3867,0.3981\} & \{0.5942,0.6057,0.6967\} \\
\{0.2736,0.2736,0.2658\} & \{0.5\} & \{0.4902,0.5000,0.5120\} & \{0.7168,0.7264,0.7418\} \\
\{0.6133,0.6133,0.6019\} & \{0.5098,0.50000,4880\} & \{0.5\} & \{0.2882,0.3867,0.4853\} \\
\{0.4058,0.3943,0.3033\} & \{0.2832,0.2736,0.2582\} & \{0.7118,0.6133,0.5147\} & \{0.5\}
\end{array}\right)
$$

and $C I\left({ }^{m} H_{3}\right)=0.1211=\overline{C I}\left(H_{3}\right)$. Therefore, ${ }^{m} H_{3}$ is of the acceptable consistency.

Using the Grayscale, created by the Matlab Drawing toolbar, to present the hesitant fuzzy preference relations with different consistency levels in Example 3.10, we can get Fig. 3.1 (Zhu et al. 2013b) which presents the inconsistent hesitant fuzzy preference relation $H_{3}$, the modified hesitant fuzzy preference relation ${ }^{m} H_{3}$ with the acceptable consistency, and the consistent hesitant fuzzy preference relation $\bar{H}_{3}$. By observational analyses, the more consistency of the hesitant fuzzy preference relation, the more uniform of distribution on grayscale.


Fig. 3.1. Grayscale of $H_{3},{ }^{m} H_{3}$ and $\bar{H}_{3}$


Fig. 3.2. Area of $H_{3},{ }^{m} H_{3}$ and $\bar{H}_{3}$

Besides of the Grayscale, we also can use the area to present the three hesitant fuzzy preference relations in Fig. 3.2 (Zhu et al. 2013b), in which the consistent hesitant fuzzy preference relation performs more regular with respect to the areas in different colors than the inconsistent hesitant fuzzy preference relations. So the difference among the hesitant fuzzy preference relations which have different consistency levels can be recognized somehow in the two figures.

For the hesitant fuzzy preference relations with the acceptable consistency, which ensures that the DMs provide almost consistent preferences, we now use the hesitant aggregation operators to aggregate the preferences in the hesitant fuzzy preference relations so as to derive the priorities to rank the alternatives.

For a set of alternatives, $A=\left\{A_{1}, A_{2} \ldots, A_{n}\right\}$, and a hesitant fuzzy preference relation $H$ constructed by the decision group. Following the $\beta$-normalization, we now develop an algorithm to deal with the prioritization problem (Zhu et al. 2013b):

## (Algorithm 3.7)

Step 1. Using Algorithm 3.6 to obtain the modified hesitant fuzzy preference relation ${ }^{m} H$ with the acceptable consistency.

Step 2. Using the hesitant aggregation operators to aggregate preferences in ${ }^{m} H$.

Step 3. Ranking all the alternatives and selecting the best one according to the aggregation results.

Step 4. End.

Example 3.11 (Zhu et al. 2013b). Suppose that a constructed hesitant fuzzy preference relation $H_{4}$ associated with a group of DMs on four alternatives $A=\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\}$ is shown as:

$$
H_{4}=\left(\begin{array}{cccc}
\{0.5\} & \{0.2,0.3\} & \{0.4,0.5,0.6\} & \{0.8,0.9\} \\
\{0.7,0.8\} & \{0.5\} & \{0.5\} & \{0.3,0.4\} \\
\{0.6,0.5,0.4\} & \{0.5\} & \{0.5\} & \{0.5,0.6,0.7\} \\
\{0.2,0.1\} & \{0.7,0.6\} & \{0.3,0.4,0.5\} & \{0.5\}
\end{array}\right)
$$

Step 1. Using Algorithm 3.6, we get the optimal $\varsigma=1$, and the modified normalized hesitant fuzzy preference relation ${ }^{m} H_{4}^{N}$ as:

$$
\left.\begin{array}{rl}
{ }^{m} H_{4}^{N}= & \left\{\begin{array}{cc}
\{0.5\} & \{0.2649,0.3643,0.3579\} \\
\{0.7351,0.6357,0.6421\} & \{0.5\} \\
\{0.5734,0.4841,0.4183\} & \{0.5000,0.4977,0.4994\} \\
\{0.2317,0.1516,0.1367\} & \{0.6203,0.5294,0.5356\}
\end{array}\right. \\
& \{0.4266,0.5159,0.5817\} \\
\{0.7683,0.8484,0.8633\} \\
& \{0.5000,0.5023,0.5006\} \\
\{0.3797,0.4706,0.4644\} \\
\{0.5\} & \{0.5296,0.6158,0.6839\} \\
\{0.4704,0.3842,0.3161\} & \{0.5\}
\end{array}\right)
$$

Step 2. Using the HFA (Eq.(1.33)) and HFG (Eq.(1.35)) operators, respectively, to aggregate the $i$ th line of preferences in ${ }^{m} h_{i j}(j=1,2,3,4)$ and get the overall performance value ${ }^{m} h_{i}$ corresponding to the alternative $A_{i}$.
Step 3. According to Definition 1.2, we can get the score values $s\left({ }^{m} h_{i}\right)(i=1,2,3,4)$ of the alternatives shown in Table 3.8 (Zhu et al. 2013b)

Table 3.8. Score values and the ranking of alternatives

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | Rankings |
| :---: | :---: | :---: | :---: | :---: | :---: |
| HFA | 0.5870 | 0.5377 | 0.5298 | 0.4247 | $A_{1} \succ A_{2} \succ A_{3} \succ A_{4}$ |
| HFG | 0.5008 | 0.5203 | 0.5217 | 0.3683 | $A_{3} \succ A_{2} \succ A_{1} \succ A_{4}$ |

By the ranking results, $A_{1}$ and $A_{3}$ are the best alternatives in accordance with the HFA and HFG operators, respectively. That is because the HFA operator pays more attention to the number of arguments. However, the HFG operator focuses on the average of arguments, and the smaller deviation between arguments, the better the results.

Furthermore, if we deal with this problem based on the $\alpha$-normalization, we can obtain a reduced hesitant fuzzy preference relation $H_{3}^{\prime}$, denoted by

$$
H_{3}^{\prime}=\left(\begin{array}{llll}
0.5 & 0.3 & 0.4 & 0.8 \\
0.7 & 0.5 & 0.5 & 0.4 \\
0.6 & 0.5 & 0.5 & 0.7 \\
0.2 & 0.6 & 0.3 & 0.5
\end{array}\right)
$$

and the priority vector of the alternatives is $w_{H_{3}}=(0.3818,0.2545,0.2545,0.1091)^{\mathrm{T}}$. So the ranking of the alternatives is $A_{1} \succ A_{2} \sim A_{3} \succ A_{4}$, which cannot distinguish $A_{2}$ and $A_{3}$. But we have $A_{2} \succ A_{3}$ according to the ranking result in Table 2.8 based on the HFA operator. Therefore, the advantage of using the $\beta$-normalization to deal with hesitant fuzzy preference relations is that all preferences provided by the decision group can be taken into account, which is helpful to make better decisions. The $\beta$-normalization can be considered as a complementary principle to the $\alpha$ - normalization if the hesitant goal programming model based on the $\alpha$-normalization cannot produce satisfying results.

Since the proposed priority methods are specially used for hesitant fuzzy preference relations, and we find no any previous works concentrating on the decision making problems using hesitant fuzzy preference relations, it's not easy to carry on with comparative illustrations. However, our models can be considered as extensions of some existing models dealing with fuzzy preference relations with multiplicative consistency, such as the goal programming models proposed by Xu (2004a). We now take an incomplete fuzzy preference relation from Xu (2004a), and use the model (M-3.3) to deal with it.

Example 3.12 (Zhu et al. 2013b). For a decision making problem with a set of alternatives $A=\left\{A_{1}, A_{2}, \ldots, A_{6}\right\}$. The DM provides his/her preferences over paired comparisons of the alternatives so as to construct an incomplete fuzzy preference relation as follows:

$$
U=\left(\begin{array}{cccccc}
0.5 & 0.4 & - & 0.3 & 0.8 & 0.3 \\
0.6 & 0.5 & 0.6 & 0.5 & - & 0.4 \\
- & 0.4 & 0.5 & 0.3 & 0.6 & - \\
0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8 \\
0.2 & - & 0.4 & 0.6 & 0.5 & 0.7 \\
0.7 & 0.6 & - & 0.2 & 0.3 & 0.5
\end{array}\right)
$$

According to the model (M-3.3), we have Eq.(3.92) as follows:

$$
\begin{align*}
& \min F=\sum_{i=1}^{6} \sum_{j=1, i \neq j}^{6}\left(d_{i j}^{+}+d_{i j}^{-}\right) \\
& \text {s.t. }\left\{\begin{array}{l}
0.6 w_{1}-0.4 w_{2}-d_{12}^{+}+d_{12}^{-}=0,0.7 w_{1}-0.3 w_{4}-d_{14}^{+}+d_{14}^{-}=0 \\
0.2 w_{1}-0.8 w_{5}-d_{15}^{+}+d_{15}^{-}=0,0.7 w_{1}-0.3 w_{6}-d_{16}^{+}+d_{16}^{-}=0, \\
0.4 w_{2}-0.6 w_{3}-d_{23}^{+}+d_{23}^{-}=0,0.5 w_{1}-0.5 w_{6}-d_{24}^{+}+d_{24}^{-}=0 \\
0.6 w_{2}-0.4 w_{6}-d_{26}^{+}+d_{26}^{-}=0,0.7 w_{3}-0.3 w_{4}-d_{34}^{+}+d_{34}^{-}=0 \\
0.4 w_{3}-0.6 w_{5}-d_{35}^{+}+d_{35}^{-}=0,0.6 w_{4}-0.4 w_{5}-d_{45}^{+}+d_{45}^{-}=0 \\
0.2 w_{4}-0.8 w_{6}-d_{46}^{+}+d_{46}^{-}=0,0.3 w_{5}-0.7 w_{6}-d_{56}^{+}+d_{56}^{-}=0 \\
\sum_{i=1}^{6} w_{i}=1, w_{i}>0, d_{i j}^{+}, d_{i j}^{-} \geq 0, i, j=1,2, \ldots, 6, i \neq j .
\end{array}\right. \tag{3.92}
\end{align*}
$$

By solving Eq.(3.92), we can obtain the following results:

$$
\left\{\begin{aligned}
w_{U} & =(0.1412,0.2118,0.1412,0.3294,0.0941,0.0824) \\
d_{12}^{+} & =d_{12}^{-}=d_{14}^{+}=d_{14}^{-}=d_{23}^{+}=d_{23}^{-}=d_{34}^{+}=d_{34}^{-}=d_{35}^{+}=d_{35}^{-} \\
& =d_{46}^{+}=d_{46}^{-}=0 \\
d_{15}^{+} & =0, d_{15}^{-}=0.0471, d_{16}^{+}=0.0741, d_{16}^{-}=0 \\
d_{24}^{+} & =0, d_{24}^{-}=0.0588, d_{26}^{+}=0.0941, d_{26}^{-}=0 \\
d_{45}^{+} & =0.1600, d_{45}^{-}=0, d_{56}^{+}=0, d_{56}^{-}=0.0294
\end{aligned}\right.
$$

So $A_{4}$ is the best alternative, which is the same as the result of Xu (2004a).

### 3.6 Deriving Priorities in AHP-Hesitant Group Decision Making

Analytic Hierarchy Process (AHP) (Saaty 1977, 1980, 1989) is one of the most popular and powerful techniques for decision making. AHP is built on a human being's intrinsic ability to structure his perceptions or his ideas hierarchically, compares pairs of similar things against a given criterion or a common property, and judges the intensity of the importance of one thing over the other (Ernest and Kirti 1998). It has been widely used for MADM problems to rank, select, evaluate and benchmark decision alternatives (Golden et al. 1989; Vaidya and Kumar 2006).

In conventional AHP, Saaty (1980) proposed four steps: (1) Modeling; (2) Valuation; (3) Prioritization; (4) Synthesis. The first of these steps involves the construction of a hierarchy at different levels of criteria, sub-criteria and alternatives. The top level of hierarchy represents the goal concerned in the problem, while the criteria, sub-criteria and alternatives are placed in the remaining levels.

The valuation step incorporates the individual judgments that reflect the relative importance of elements at a level of hierarchy through pairwise comparison judgments, which are described as various preference relations, such as multiplicative preference relations (Saaty 1980), interval multiplicative preference relations, fuzzy preference relations and interval fuzzy preference relations, and so on (see, e.g. Orlovski 1978; Tanino 1984; Xu and Wei 1999; Yager 2004; Xu 2004b; Chandran et al. 2005; Herrera et al. 2005).

The third step considers the local and global priorities of each element of hierarchy. According to some common prioritization methods, the local priorities can be obtained, such as the eigenvector method (Saaty 1977), the logarithmic least squares method (Crawford and Williams 1985), and the logarithmic goal programming method (Bryson 1995). The eigenvector method and the logarithmic goal programming method are the most used two methods in practice, and have desirable properties respectively. Many researchers have studied on the comparisons between these methods so as to choose one with the best performance (Barzilai 1997; Zahedi 1986; Saaty 1990). Golany and Kress (1993) concluded that there is no prioritization method that is superior to the other ones in all cases. Every method has its own advantages and drawbacks, and we should choose a method according to the objective of the analysis. The global priorities of any level of hierarchy are calculated by applying the hierarchical comparison principle (Saaty 1980) and reflect the priority of any level with respect to the main goal.

In the final step, some aggregation procedures (the weighted arithmetic average and the geometric mean are the two most common ones) are used to synthesize the global priorities of alternatives so as to obtain the final priorities of the alternatives. Consequently, the ranking results of the alternatives can be obtained with these final priorities for decision making.

The desirable characteristics of flexibility and adaptability of AHP permit its use in group decision making. Moreno-Jiménez et al. (2002) identified three
possibilities: (1) Group decision making; (2) Negotiated decision making; (3) Systemic decision making. In the first case, the individuals act jointly in pursuit of a common decision. In the second case, each individual solves the problem independently, and the agreement and disagreement zones are analyzed in order to reach a consensus. Finally, in the third case, each individual solves the problem individually, and a tolerance principle is used to look for a way of integrating all the positions.

Since the judgments provided by individuals are not perfect in most cases, AHP specially allows the measurement of consistency or inconsistency degree in the preference relation, which is a desirable characteristic in contrast to other multi-criteria decision techniques. Saaty (1977) developed a consistency index (CI) and a consistency ratio ( CR ) to measure the level of inconsistency. If $C R<0.1$, then the preference relation is considered to be of acceptable consistency (otherwise, unacceptable consistency), and AHP may not yield meaningful results. Many methods concentrate on the improvement of preference relations with unacceptable consistency (Xu and Wei 1999; Cao et al. 2008). Aguarón and Moreno-Jiménez (2003) proposed the use of the geometric consistency index (Crawford and Williams 1985) with the row geometric mean prioritization procedure.

In AHP-based group decision making (AHP-GDM), there are two systems of approaching group decision making with a view to aggregating the individual judgments into a group judgment. Forman and Peniwati (1998) showed that depending on whether the relative importance of individuals in the group assumed to be equal, or incorporated in the aggregation process, they are specified as: (1) Aggregating individual judgments (AIJ); (2) Aggregating individual priorities (AIP). In AIJ, individual identities are lost with every stage of a synergistic aggregation of individual judgments, and a synthesis of hierarchy produces the group's priorities. At each level, a merging process occurs step by step. A common hierarchy should be agreed on first by the group working together, and then the relative importance of the criteria, after that, the merging process occurs continually at the judgment level. In such a way, the individuals act in concert and pool their judgments, and they become a new "individual" and behave like one. Three main prioritization procedures are utilized to aggregate individual judgments: consensus, voting, and aggregated methods including the weighted arithmetic average and the geometric mean (Saaty 1989). In the consensus and voting approaches, the group agrees upon all the comparison judgments. If the group is unwilling or unable to vote or cannot achieve a consensus, then individual judgments are aggregated into an aggregated group judgment. Aczel and Saaty (1983) showed that a geometric mean rather than the weighted arithmetic average in AIJ must be used for aggregation of individual judgments. The three prioritization procedures considering all individuals' judgments naturally result in a compromise group solution.

In AIP, the priorities are obtained from each individual hierarchy, and then aggregated into final group priorities. For individuals, they can act in their own right, with different value systems or different hierarchies. We are just concerned about each individual's resulting alternative priorities. In this situation, the Pareto
principle (It says that given two alternatives $A_{1}$ and $A_{2}$, if each member of a group of individuals prefers $A_{1}$ to $A_{2}$, then the group must prefer $A_{1}$ to $A_{2}$ ) is satisfied. Forman and Peniwati (1998) stated that both the weighted arithmetic average and the geometric mean could be used to aggregate the individual's priorities.

In AHP-GDM, suppose that individuals in the group are required to provide comparison judgments in accordance with Saaty's fundamental scale (Saaty 1980, 1994) with respect to a level of hierarchy, which reflect the relative importance degrees of elements. Due to their different levels of expertise and the complexity of the problem, individuals consequently provide different judgments. Traditional AHP uses the prioritization procedures to aggregate individual judgments resulting in a compromise result. However, in practice, a best solution is naturally the ultimate goal in decision making rather than a compromise result. Therefore, Zhu and Xu (2013b) utilized the desirable characteristic that the measurement of consistency of AHP, to relinquish some judgments for the good of the group so as to find the best solution which has the highest group consensus degree. The premise condition of this approach is that the group acts together as a unit, hesitates about several possible judgments. According to Moreno-Jiménez et al. (2002), this should be a group decision making problem but without a common decision due to several possible values. We consider this as AHP-hesitant group decision making (AHP-HGDM) as an extension of AHP-GDM.

In AHP-HGDM, the group does not have some degree of uncertainty represented by interval values or fuzzy values, but hesitates about several possible judgments. We do not aggregate individuals' judgments, but try to select a group judgment among all possible ones so as to obtain the best group solution under limited options. Since the group acts as a unit, each possible judgment from the group can be considered as a final choice, we confront with a selection problem rather than aggregation, consensus or voting. Existing researches on AHP-GDM mainly concentrate on these prioritization procedures based on comparison judgments represented by crisp values (Saaty 1980, 1994), interval values (Saaty and Vargas 1987; Arbel 1989; Zahir 1991; Sugihara and Tanaka 2001; Sugihara et al. 2004), or triangular fuzzy values (Van Laarhoven and Pedrycz 1983; Chang 1996; Zhu et al. 1999). But AHP-HGDM is characterized by a selection process based on some possible comparison judgments.

AHP-HGDM also involves the four steps of Saaty's AHP. The first and last steps, i.e. modeling and synthesis, obey Saaty's approach. The second step, valuation, allows the group to give several possible comparison judgments so as to construct a preference relation defined as hesitant multiplicative preference relations. The prioritization step, deriving priorities from the hesitant multiplicative preference relation, is the major constituent of AHP-HGDM.

The eigenvector method and the logarithmic least squares method are the two most known prioritization methods, where the eigenvector method is the original Saaty's method. Assume a multiplicative preference relation $B=\left(b_{i j}\right)_{n \times n} \in \mathfrak{R}^{n \times n}$, and a priority vector $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$ derived from $B$, the eigenvector
method is based on the fact that small perturbations of the elements $b_{i j}$ from the perfect ratios $\frac{w_{i}}{w_{j}}$ lead to small perturbations of the eigenvalues of $B$ around the eigenvalues of a consistent preference relation of $B$. The eigenvector method is not suitable for AHP-HGDM which has difficulty in obtaining the eigenvalues from a hesitant multiplicative preference relation. The logarithmic least squares method, also known as a geometric mean method, is based on the assumption that the elements of the priority vector $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$ should best satisfy the property $b_{i j} \approx \frac{w_{i}}{w_{j}}$. This priority assessment is formulated as a constrained optimization problem:

$$
\begin{align*}
& \min \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\ln b_{i j}-\ln w_{i}+\ln w_{j}\right)^{2}  \tag{3.93}\\
& \text { s.t. } \sum_{i=1}^{n} w_{i}=1, w_{i} \geq 0, i=1,2, \ldots, n
\end{align*}
$$

This method tries to find a solution satisfying "all constraints" and results in a compromise result, which is also not suitable for the selection problem in AHP-HGDM.

As an alternative method of the eigenvector method and the logarithmic least squares method, Mikhailov (2000) developed a fuzzy programming method, which transforms the prioritization problem into a fuzzy programming problem that can easily be solved as a standard linear program. He compared the fuzzy programming method with the eigenvector method, the logarithmic least squares method and the logarithmic goal programming method et al., and showed that the fuzzy programming method outperforms some of the existing methods, especially in highly inconsistent cases. The fuzzy programming method has some attractive properties, such as simplicity of the computation algorithm, good precision and rank preservation. It easily deals with missing judgments and provides a natural consensus indicator for measuring the satisfaction degree of the group solution.

Based on the advantages of fuzzy programming method, Zhu and Xu (2013b) developed a hesitant fuzzy programming method as a prioritization method to derive priorities from hesitant multiplicative preference relation in AHP-HGDM.

### 3.6.1 Description of the Prioritization Method

In this subsection, we give a description of the resolution process of the hesitant fuzzy programming method. The hesitant fuzzy programming method is a prioritization method used for deriving priorities from hesitant multiplicative
preference relations in AHP-HGDM. The hesitant multiplicative preference relation is constructed by group pairwise comparison judgments, which is stated as follows (Zhu and Xu 2013b):

Let $A=\left\{A_{1}, A_{2}, \cdots, A_{n}\right\}(n \geq 2)$ be a finite set of elements at a level of hierarchy in AHP-HGDM. The preference information of a group on $A$ is described by a hesitant multiplicative preference relation $R=\left(r_{i j}\right)_{n \times n} \subset A \times A$, where $r_{i j}=\left\{r_{i j}^{q} \mid q=1,2, \cdots, l_{r_{i j}}\right\}\left(l_{r_{i j}}\right.$ is the number of comparison judgments in $\left.r_{i j}\right) . r_{i j}$ denotes the comparison judgment element $A_{i}$ with respect to $A_{j}$. The measurement of $r_{i j}$ is described using a ratio scale and in particular, as given by Saaty (1980), $\quad r_{i j} \subset\left\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1,2, \ldots, 7,8,9\right\}: \quad r_{i j}=\{1\} \quad\left(r_{j i}=\{1\}\right)$ denotes the indifference between $A_{i}$ and $A_{j}, r_{i j}=\{9\}\left(r_{j i}=\left\{\frac{1}{9}\right\}\right)$ denotes that $A_{i}$ is unanimously preferred to $A_{j}$, and $r_{i j} \in\left\{\frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1,2, \ldots, 7,8\right\}$ denotes the intermediate evaluations. This hesitant multiplicative preference relation satisfies the conditions as shown in Eq.(3.6).

If there are missing elements in the hesitant multiplicative preference relation, then it reduces to an incomplete hesitant multiplicative preference relation; If $l_{r_{i j}}=1$ for all $i, j=1,2 \ldots, n$, which means that the group provides unique comparison judgments for all $r_{i j}(i, j=1,2 \ldots, n)$, then $R=\left(r_{i j}\right)_{n \times n}$ reduces to a multiplicative preference relation $B=\left(b_{i j}\right)_{n \times n} . B$ is perfectly consistent if $b_{i k} b_{k j}=b_{i j}(i, j, k \in\{1,2, \ldots, n\})$. It's clear that $R$ can reduce to various multiplicative preference relations depending on the selection of $r_{i j}^{q}$ from $r_{i j}(i, j=1,2 \ldots, n)$.

The hesitant fuzzy programming method is a linear programming method combined with a selection process. According to the hesitant fuzzy programming method, a number of possible group solutions can be obtained from a hesitant multiplicative preference relation. In such a case, we should select one that performs better than others. The hesitant fuzzy programming method provides a group consensus index ( $G C I$ ) that measures the satisfaction or dissatisfaction degree for a possible group solution, that is, the higher $G C I$, the better of the group solution. Thus, we can select out a best one with the highest value as a final group solution among all possible ones by the $G C I$.

In most cases, by solving the linear programming model and the selection process, we can get a unique final group solution with the highest $G C I$, however, sometimes, we need further decisions due to several group solutions with the same highest $G C I$. We refer to an anonymous feedback mechanism in this situation to help the group members revise their judgments. The resolution process of the hesitant fuzzy programming method at a level of hierarchy in AHP-HGDM can be graphically represented in Fig. 3.3 (Zhu and Xu 2013b) (in which HMPR denotes hesitant multiplicative preference relation, and HFPM denotes hesitant fuzzy programming method), which includes two steps and a feedback mechanism (Zhu and Xu 2013b):

Step 1. Use the hesitant fuzzy programming model to obtain all possible group solutions with the GCI from the hesitant multiplcative preference relation; If there is a unique one with the highest $G C I$, then select it as the final group solution and go to Step 2; If not, then return suggestions to the group to help them revise their judgment so as to construct a new hesitant multiplcative preference relation, and then repeat Step 1 until we have the final group solution.

Step 2. The priorities of elements are got according to the final group solution.


Fig. 3.3. Resolution process of the hesitant fuzzy programming method

### 3.6.2 Hesitant Fuzzy Programming Method

Zhu and Xu (2013b) gave a geometric representation of the priorities derivation problem based on the hesitant multiplcative preference relation. Consider a group,
comparing pairwisely $n$ elements at the same level of hierarchy in AHP-HGDM, constructing a hesitant multiplcative preference relation, $R=\left(r_{i j}\right)_{n \times n}$, where $r_{i j}=\left\{r_{i j}^{q} \mid q=1,2, \cdots, l_{r_{i j}}\right\}$. Suppose that the priority vector is $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}}$, where $\sum_{i=1}^{n} w_{i}=1, \quad w_{i} \geq 0, \quad i=1,2, \ldots, n$. The comparison judgment $r_{i j}$ is the estimation of the ratio $\frac{w_{i}}{w_{j}}$. Suppose $r_{i j}^{q} w_{j}-w_{i}=0$ for all upper triangular elements in $R$, i.e., $i=1,2, \ldots, n-1$, $j=2,3, \ldots, n, j>i, q=1,2, \ldots, l_{q_{i j}}$, they can be represented as a set of linear equalities:

$$
\begin{equation*}
R_{i j} w=\left\{R_{i j}^{q} w=0 \mid q=1,2, \ldots, l_{r_{i j}}\right\}, i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i \tag{3.94}
\end{equation*}
$$

where $R_{i j}^{q} w=0$ defines a $i, j, q$ grouped hyperplane in the $n$-dimensional priority space, denoted by

$$
\begin{equation*}
G_{i j}^{q}(w)=\left\{w^{q} \mid R_{i j}^{q} w=0\right\} \tag{3.95}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
G_{i j}(w)=\left\{G_{i j}^{q}(w) \mid q=1,2, \ldots, l_{r_{i j}}\right\}, i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i \tag{3.96}
\end{equation*}
$$

Let $\quad G_{o}(w)=\left\{w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{\mathrm{T}} \mid w_{1}+w_{2}+\ldots+w_{n}=1\right\}$ be the simplex hyperplane. Since the priority vector must lie on $G_{o}(w)$, we should consider the intersections of $G_{i j}^{q}(w)$ and $G_{o}(w)$. Let $L_{i j}^{q}(w)$ indicate the $i, j, q$ grouped hyperline defined by the intersection between $G_{i j}^{q}(w)$ and $G_{o}(w)$, where

$$
\begin{equation*}
L_{i j}^{q}(w)=G_{i j}^{q}(w) \bigcap G_{o}(w) \tag{3.97}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
L_{i j}(w)=\left\{L_{i j}^{q}(w) \mid q=1,2, \ldots, l_{r_{i j}}\right\}, i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i \tag{3.98}
\end{equation*}
$$

Then the intersection of the $i, j, q$ grouped hyperlines $L_{i j}^{q}(w)$ for all $i$ and $j$ on $G_{o}(w)$ is represented as:

$$
\begin{equation*}
L^{q}(w)=\bigcap_{i} \bigcap_{j} L_{i j}^{q}(w) i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i \tag{3.99}
\end{equation*}
$$

which indicates the $q$ th solution of priority vector at a level of hierarchy in AHP-HGDM, where $L_{i j}^{q}(w) \in L_{i j}(w), \quad q=1,2, \ldots, l_{r}, \quad l_{r}=\prod_{i, j} l_{r_{i j}}$. Consequently, we have a set of all possible solutions as:

$$
\begin{equation*}
L(w)=\bigcup_{L^{q}(w) \in L(w)}=\left\{L^{q}(w) \mid q=1,2, \ldots, l_{r}\right\} \tag{3.100}
\end{equation*}
$$

Each solution derived from the prioritization problem, $L^{q}(w)$, corresponds to a multiplicative preference relation as a selection result from the multiplicative preference relation hesitant multiplicative preference relation, $R=\left(r_{i j}\right)_{n \times n}$. If there exists a consistent multiplicative preference relation, $B=\left(b_{i j}\right)_{n \times n}$, selected from $R$, then $L^{q}(w)$ on the simplex hyperplane is not empty and contains only one point which gives the solution of $R_{i j}^{q} w=0$ for all $i$ and $j$ under the conditions $\sum_{i=1}^{n} w_{i}=1, w_{i} \geq 0, i=1,2, \ldots, n$. If we have several solutions, which means that there are several consistent multiplicative preference relations, selected from $R$, then we should refer to the feedback mechanism discussed in Subsection 3.6.1.

In practice, the group tries to achieve a consensus so as to provide constructive suggestions. Thus, it's reasonable to try to find a solution that satisfies all judgments "as much as possible" at a level of hierarchy in AHP-HGDM. It means that a priority vector satisfies all equalities approximately, represented as a system of fuzzy equalities:

$$
\begin{equation*}
R_{i j}^{q} w \cong 0, \quad i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i \tag{3.101}
\end{equation*}
$$

where $R_{i j}^{q} w \in R_{i j} w, R_{i j}^{q} w=w_{i}-r_{i j}^{q} w_{j}$, the symbol " $\cong$ " denotes the statement "approximately equal to".

If the ratio of priorities $\frac{w_{i}}{w_{j}}$ equals $r_{i j}^{q}$, the degree of the group satisfaction is equal to one; Otherwise, the degree of satisfaction should decrease to some deviation limits. Therefore, we can use the fuzzy sets to describe such conditions. Assume that the satisfaction for the hyperplane $R_{i j}^{q} w=0$ is measured by a linear membership function, $m_{i j}^{q}(w)$, and $\mathcal{E}$ is the deviation parameter determined by the group. The membership function is linearly increasing over the interval $[-\infty, 0]$, and linearly decreasing over the interval $[0,+\infty]$. In the case $R_{i j}^{q} w=0$, the membership $m_{i j}^{q}(w)=1$ indicates the complete satisfaction; In the case $R_{i j}^{q} w \in[-\mathcal{E}, \mathcal{\varepsilon}]$, the membership $0 \leq m_{i j}^{q}(w) \leq 1$ indicates partial satisfaction; In the case $R_{i j}^{q} w \notin[-\varepsilon, \varepsilon], m_{i j}^{q}(w)<0$ indicates dissatisfaction. The membership function is defined as a $L$-fuzzy sets taking values in the range $[-\infty, 1]$, which can measure not only the degree of satisfaction, but also the degree of dissatisfaction.

Therefore, the fuzzy equality Eq. (3.101), on the simplex hyperplane $G_{o}(w)$, can be characterized by a linear convex membership function:

$$
m_{i j}^{q}\left(R_{i j}^{q} w\right)= \begin{cases}1-\frac{R_{i j}^{q} w}{\varepsilon}, & \text { if } R_{i j}^{q} w \geq 0  \tag{3.102}\\ 1+\frac{R_{i j}^{q} w}{\varepsilon}, & \text { if } R_{i j}^{q} w<0\end{cases}
$$

where $R_{i j}^{q} w \in R_{i j} w$, and the deviation parameter $\mathcal{E}$ is determined by the group.
The intersection of the $i, j, q$ grouped membership function $m_{i j}^{q}(w)$ for all $i$ and $j$, is defined as the $q$ th fuzzy feasible area $\tilde{L}^{q}(w)$ which is also a membership function:

$$
\begin{equation*}
m_{\tilde{L}^{q}}(w)=\min \left\{m_{i j}^{q}(w) \mid i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i\right\} \tag{3.103}
\end{equation*}
$$

If $\mathcal{E}$ is large enough, then $\tilde{L}^{q}(w)$ on $G_{o}(w)$ is not empty; If $\mathcal{E}$ is small or judgments given by the group are very inconsistent, then the membership function can take negative values according to the $L$-fuzzy sets.

On the basis of the results obtained by Dubois and Fortemps (1999) on the best solutions to the max-min optimization problems with convex domains, Eq.(3.103) is a convex fuzzy set, and we can obtain a unique best solution of the priority vector $w=\left(w_{1}, \ldots, w_{n}\right)^{\mathrm{T}}$ with a maximum degree of membership for the $k$ th fuzzy feasible area defined as $\lambda^{(q)}$ (Zhu and Xu 2013b):

$$
\begin{equation*}
\lambda^{(q)}=m_{\tilde{L}^{q}}(w)=\max \min \left\{m_{i j}^{q}(w) \mid i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i\right\} \tag{3.104}
\end{equation*}
$$

Proof. Let the solutions of priority vector obeying the membership function Eq.(3.102) be the domain O . Suppose that two best solutions $w^{1}$ and $w^{2}$ in O satisfying $m_{\tilde{L}^{q}}\left(w^{1}\right)=m_{\tilde{L}^{q}}\left(w^{2}\right)=\lambda^{(q)}$, and their convex combination is $w=\alpha w^{1}+(1-\alpha) w^{2} \quad(0<\alpha<1)$. Thus, we have

$$
\begin{equation*}
m_{\tilde{L}^{9}}(w)>\min \left\{m_{\tilde{L}^{a}}\left(w^{1}\right), m_{\tilde{L}^{a}}\left(w^{2}\right)\right\} \tag{3.105}
\end{equation*}
$$

thus

$$
\begin{equation*}
m_{\tilde{L}^{q}}(w)>\lambda^{(q)} \tag{3.106}
\end{equation*}
$$

Therefore, $w^{1}$ and $w^{2}$ are not the best solutions of priority vector. In this application, there exists only one unique solution which provides the priority vector coherent with all comparison judgments for the fuzzy feasible area $\tilde{L}^{q}(w)$.

For $\tilde{L}^{q}(w)$, the formulation of the max-min problem given by Eq.(3.104) is the fuzzy programming method developed by Mikhailov (2000), which can be stated as follows:

$$
\begin{cases}\max & \lambda^{(q)}  \tag{3.107}\\ \text { s.t. } & \lambda^{(q)} \leq m_{i j}^{q}\left(R_{i j}^{q} w\right)\end{cases}
$$

As introduced by Mikhailov (2000), Eq.(3.107) is similar to the maximizing decision rule in the decision making in fuzzy environment with fuzzy goals and fuzzy constraints, proposed by Bellman and Zadeh (1970). This prioritization problem can be seen as the fuzzy linear programming problem studied by Zimmermann (1976). According to Eq.(3.102), Eq.(3.107) can be represented as the following linear programming problem (Zhu and Xu 2013b):

$$
\begin{cases}\max & \lambda^{(q)}  \tag{3.108}\\ \text { s.t. } & \varepsilon \lambda^{(q)}+R_{i j}^{q} w \leq \varepsilon \\ & \varepsilon \lambda^{(q)}-R_{i j}^{q} w \leq \varepsilon \\ & i=1,2, \ldots, n-1, j=2,3, \ldots, n, j>i, \\ & \sum_{i=1}^{n} w_{i}=1, w_{i}>0, i=1,2, \ldots, n .\end{cases}
$$

The group solution to Eq.(3.108) is a vector $\bar{w}^{(q)}=\left(\lambda^{(q)},\left(w^{(q)}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$, and in the process of solving Eq.(3.108), we can obtain a corresponding multiplicative preference relation $B^{(q)}$ with $\bar{w}^{(q)}$ as a selection result from $R$. The first component of the group solution, $\lambda^{(q)}$, indicates the maximum membership degree. By Eqs.(3.102) and (3.104), we have $\lambda^{(q)}=m_{\tilde{L}^{q}}(w) \leq m_{i j}^{q}\left(R_{i j}^{q} w\right)$. Since the membership function $m_{i j}^{q}\left(R_{i j}^{q} w\right)$ is defined as a $L$-fuzzy set taking values in the range $[-\infty, 1]$, thus $\lambda^{(q)} \leq 1$. If $\lambda^{(q)}=1$, then $B^{(q)}$ is consistent; If $0 \leq \lambda^{(q)}<1$, then $B^{(q)}$ is approximately consistent within the tolerance parameter $\mathcal{E}$; If $\lambda^{(q)}<0$, then $B^{(q)}$ is inconsistent. Therefore, $\lambda^{(q)}$ can be considered as a $G C I$ that measures the satisfaction or dissatisfaction degree for a possible group solution, the larger the $G C I$, the better consensus degree of the group solution.

The second component of the group solution is a priority vector $w^{(q)}$ under $\lambda^{(q)}$. For all fuzzy feasible areas $\tilde{L}^{q}(w), q=1,2, \ldots, l_{r}$, by solving Eq.(3.108), we have $\bar{w}=\left\{\bar{w}^{(q)} \mid q=1,2, \ldots, l_{r}\right\}$. In order to obtain a final group solution, we should select out the largest $\lambda^{*}$ that reflects the maximum degree of satisfaction, denoted by

$$
\begin{equation*}
\lambda^{*}=\max \left\{\lambda^{(q)} \mid q=1,2, \ldots, l_{r}\right\} \tag{3.109}
\end{equation*}
$$

For each $\lambda^{*}$, we have the final group solution $\bar{w}^{*}=\left(\lambda^{*},\left(w^{*}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$, and a corresponding multiplicative preference relation $B^{*}$. If we have several $\lambda^{(q)}$ that have the same greatest value of $G C I$, but under each $\lambda^{(q)}$, there is a different $w^{(q)}$, then in such a case, we should refer to the anonymous feedback mechanism introduced in Subsection 3.1.

The key of the anonymous feedback mechanism is to provide suggestions to help the group revise their judgments. Assume that we have a set $\lambda=\left\{\lambda^{(q)} \mid q=1,2, \ldots, q_{0}\right\}$, where each element in the set has the same greatest value. Under $\lambda^{(q)}$, their corresponding multiplicative preference relations are $B^{(q)}\left(q=1, \ldots, q_{0}\right)$, where $B^{(q)}=\left(b_{i j}^{(q)}\right)_{n \times n}, \quad q=1, \ldots, q_{0}$. Based on $B^{(q)}\left(q=1, \ldots, q_{0}\right)$, we construct a new hesitant multiplicative preference relation $R^{\left(q_{0}\right)}=\left(r_{i j}^{\left(q_{j}\right)}\right)_{n \times n}$, where $r_{i j}^{\left(q_{0}\right)}=\left\{b_{i j}^{1}, \ldots, b_{i j}^{\left(q_{0}\right)}\right\}$, as a suggestion for the group to provide judgments within $R^{(q)}$. According to the anonymous feedback mechanism, we can find the final group solution to a level of hierarchy in AHP-HGDM.

Based on the $G C I$, we can easily select the final group solution that has the maximum degree of satisfaction without an additional computation process for the group consensus degree. The value of $\lambda^{*}$ depends on the values of the deviation parameter $\mathcal{E}$, for different $\mathcal{E}$, we have different $\lambda^{*}$. A large value of $\mathcal{E}$ can ensure non-emptiness of the global fuzzy feasible and a positive value of $\lambda^{*}$. However, for a same prioritization problem at a level of hierarchy, utilizing the hesitant fuzzy programming method, the final group solution $\bar{w}^{*}=\left(\lambda^{*},\left(w^{*}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$ always has the same priority vector $w^{*}$ although there are different $\lambda^{*}$ with respect to different $\mathcal{E}$. Therefore, $\lambda^{*}$ can be considered as an intermediate parameter used for the selection process in the hesitant fuzzy programming method, takes values within $(-\infty, 1]$. It's not necessary to set a large value of $\varepsilon$ to ensure a positive $\lambda^{*}$, we can always obtain the same priority vector in the final group solution. Without loss of generality, we can set $\varepsilon=1$ in practice.

In AHP-GDM, both the aggregation approaches, AIJ and AIP, need group synthesis and prioritization (Bryson and Joseph 1999). However, in AHP-HGDM, all the group comparison judgments are represented by the hesitant fuzzy
programming method, a synthesis procedure that combines the AIJ or AIP into group judgments is not necessary, and a final group solution with the highest satisfaction degree can be obtained directly by the hesitant fuzzy programming method. Furthermore, AIJ and AIP need full consistent comparison judgments for a set of elements. In hesitant fuzzy programming method, the group is freedom to provide comparison judgments, it means that the group can give crisp, several possible, or even ignore some comparison judgments, and the group does not need to ensure the consensus degree of their judgments, which is practical and effective in actual applications. According to comparison judgments from the group, we can construct a hesitant multiplicative preference relation or incomplete hesitant multiplicative preference relation, and it should be noted that if the group provide full crisp pairwise comparison judgments, then the hesitant multiplicative preference relation reduces to a multiplicative preference relation, and the hesitant fuzzy programming method reduces to the fuzzy programming method.

### 3.6.3 Numerical Examples

In this subsection, we give two examples. The first one illustrates the hesitant fuzzy programming method in detail. The second shows an application of the method to the water conservancy in China introduced by Zhang (2009).

Example 3.13 (Zhu and Xu 2013b). We first give a simplified example for illustrative purpose and for better understanding of the hesitant fuzzy programming method. Consider a group, comparing pairwisely four elements at a level of hierarchy in AHP-HGDM and providing the following hesitant multiplicative preference relation:

$$
R=\left[\begin{array}{cccc}
\{1\} & \{2,3\} & \{6,7\} & \{1,2,3\} \\
\left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \{2,3\} & \left\{\frac{1}{8}, \frac{1}{7}\right\} \\
\left\{\frac{1}{7}, \frac{1}{6}\right\} & \left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \left\{\frac{1}{3}\right\} \\
\left\{\frac{1}{3}, \frac{1}{2}, 1\right\} & \{7,8\} & \{3\} & \{1\}
\end{array}\right]
$$

Remark 3.1 (Zhu and Xu 2013b). Since the hesitant fuzzy programming method only takes the upper triangular elements in the hesitant multiplicative preference relation into account, we only give the upper triangular elements in the hesitant multiplicative preference relation or the incomplete hesitant multiplicative preference relation.

By solving the linear programming model Eq.(3.108), and the selection process Eq.(3.109) (accomplished by the use of the Matlab optimization toolbox), we can obtain the final group solutions $\bar{w}^{*}=\left(\lambda^{*}, w_{1}^{*}, w_{2}^{*}, w_{3}^{*}, w_{4}^{*}\right)^{\mathrm{T}}$ by the hesitant fuzzy programming method, shown in Table 3.9 (Zhu and Xu 2013b).

Table 3.9. The final group solutions for different deviation parameter $\mathcal{E}$

|  | $\varepsilon=1$ | $\varepsilon=0.5$ | $\varepsilon=0.1$ | $\varepsilon=0.05$ |
| :---: | :---: | :---: | :---: | :---: |
| $\lambda^{*}$ | 0.9423 | 0.8846 | 0.4231 | -0.1538 |
| $w_{1}^{*}$ | 0.4038 | 0.4038 | 0.4038 | 0.4038 |
| $w_{2}^{*}$ | 0.1155 | 0.1155 | 0.1155 | 0.1155 |
| $w_{3}^{*}$ | 0.0769 | 0.0769 | 0.0769 | 0.0769 |
| $w_{4}^{*}$ | 0.4038 | 0.4038 | 0.4038 | 0.4038 |

It's clear that for different $\mathcal{E}$, we have different $\lambda^{*}$, but under each $\lambda^{*}$, but have the same priority vector $w^{*}=(0.4038,0.1155,0.0769,0.4038)^{\mathrm{T}}$. As discussed in Section 3.6.2, we set $\varepsilon=1$ in the rest of subsection.

If $R$ is incomplete, shown as:

$$
R^{\prime}=\left[\begin{array}{cccc}
\{1\} & \{2,3\} & \{6,7\} & - \\
\left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \{3,2\} & - \\
\left\{\frac{1}{7}, \frac{1}{6}\right\} & \left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \{1 / 3\} \\
- & - & \{3\} & \{1\}
\end{array}\right]
$$

By the hesitant fuzzy programming method, we can obtain two group solutions that have the same maximum satisfaction degree, shown as:

$$
\begin{aligned}
& \bar{w}^{(1)}=(1.0000,0.4615,0.2308,0.0769,0.2308)^{\mathrm{T}} \\
& \bar{w}^{(2)}=(1.0000,0.5000,0.1667,0.0833,0.2500)^{\mathrm{T}}
\end{aligned}
$$

Their two corresponding multiplicative preference relations $B^{(1)}$ and $B^{(2)}$ can be obtained in the resolution process of the hesitant fuzzy programming method, given by

$$
B^{(1)}=\left[\begin{array}{cccc}
1 & 2 & 6 & - \\
\frac{1}{2} & 1 & 3 & - \\
\frac{1}{6} & \frac{1}{3} & 1 & 1 / 3 \\
- & - & 3 & 1
\end{array}\right], \quad B^{(2)}=\left[\begin{array}{cccc}
1 & 3 & 6 & - \\
\frac{1}{3} & 1 & 2 & - \\
\frac{1}{6} & \frac{1}{2} & 1 & 1 / 3 \\
- & - & 3 & 1
\end{array}\right]
$$

In this case, we refer to the anonymous feedback mechanism: We first construct the hesitant fuzzy programming method by $B^{(1)}$ and $B^{(2)}$ as follows:

$$
R^{(2)}=\left[\begin{array}{cccc}
\{1\} & \{2,3\} & \{6\} & - \\
\left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \{2,3\} & - \\
\left\{\frac{1}{6}\right\} & \left\{\frac{1}{3}, \frac{1}{2}\right\} & \{1\} & \left\{\frac{1}{3}\right\} \\
- & - & \{3\} & \{1\}
\end{array}\right]
$$

Return $B^{(2)}$ to the group as a suggestion for the group to provide further judgments within $B^{(2)}$. Assume that the group revises the judgment $r_{12}^{(2)}=\{2,3\}$ into $r_{12}^{(2)}=\{2\}$, and retains $r_{23}^{(2)}=\{2,3\}$. By the hesitant fuzzy programming method, we can get the final group solution as:

$$
\bar{w}^{*}=(1.0000,0.4615,0.2308,0.0769,0.2308)^{\mathrm{T}}=w^{(1)}
$$

In most cases of the hesitant fuzzy programming method, the final group solution is unique, especially a large number of elements at a level of hierarchy. Thus, we do not commonly need the corresponding multiplicative preference relation which is only used in the feedback mechanism.

Example 3.13 (Zhu and Xu 2013b). As one of the infrastructure industries of a national economy, water conservancy plays a great role for the sustainable development. Zhang (2009) analyzed the necessity to find out the efficient spatial allocation of water conservancy investment of capital construction (WCICC) in practical work of water conservancy in China, and used AHP-GDM combined with survey to deal with this problem. Twenty-seven qualified DMs (experts) are invited to attend a survey under the assistance provided by the Ministry of Water Resources.

The main projects of water conservancy construction include reservoir project, irrigation project, flood control and prevention project, waterlog control project, water supply project, hydropower project, and water and soil conservation project. Nine river basins in China are taken into account: Yangtse River basin, Yellow River basin, Haihe and Luanhe Rivers basin, Song and Liao Rivers basin, Inland Rivers basin, Huaihe River basin, Zhujiang River basin, South-east Rivers basin, and South-west Rivers basin, shown in Fig. 3.4 (Zhang 2009).

Zhang (2009) gave a hierarchy structure of this problem, that is, the reasonable allocation of WCICC among the nine river basins as the goal level, the seven kinds of projects as the criterion level and the nine major river basins as the scheme level. A survey form including the short-term and the medium-term circumstances for DMs (experts) is designed, which is used to collect pairwise comparison judgments from the DMs so as to judge the relative importance degree of the seven kinds of water conservancy projects and of the nine river basins in each kind of project of WCICC.


Fig. 3.4. China's nine major river basins

At each level of hierarchy, the DMs are required to provide crisp pairwise comparison judgments, and the eigenvector method is used as the prioritization method to derive priorities. Take the short-term of the water conservancy construction as an example, Zhang (2009) gave the following crisp pairwise comparison judgments of the water conservancy projects, shown in Table 3.10 (Zhu and Xu 2013b):

Table 3.10. The crisp pairwise comparison judgments of the water conservancy projects

|  | Reservoir <br> project | Irrigation <br> project | Waterlog <br> control <br> project | Flood <br> control and <br> prevention <br> project | Water <br> supply <br> project | hydropower <br> project | water and soil <br> conservation <br> project |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Reservoir project | 1 | 3 | 3 | $\frac{1}{4}$ | $\frac{1}{3}$ | 4 | 2 |
| Irrigation project |  | 1 | 1 | $\frac{1}{6}$ | $\frac{1}{5}$ | 2 | $\frac{1}{2}$ |
| Waterlog control <br> project |  | 1 | $\frac{1}{6}$ | $\frac{1}{5}$ | 2 | $\frac{1}{2}$ |  |
| Flood control and <br> prevention <br> project |  |  | 1 | 2 | 7 | 5 |  |
| Water supply <br> project |  |  |  | 1 | 6 | 4 |  |
| hydropower <br> project |  |  |  | 1 | $1 / 3$ |  |  |
| water and soil <br> conservation <br> project |  |  |  |  |  | 1 |  |

For comparison, we use the hesitant fuzzy programming method to obtain the priorities of the seven water conservancy projects, the results are shown in Table 3.11 (Zhu and Xu 2013b):

Table 3.11. The priorities of the seven projects by the hesitant fuzzy programming method and the eigenvector method

|  | Reservoir <br> project | Irrigatio <br> n project | Waterlog <br> control <br> project | Flood control <br> and prevention <br> project | Water supply <br> project | Hydro <br> power <br> project | Water and soil <br> conservation <br> project |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Priorities | $w_{1}$ | $w_{2}$ | $w_{3}$ | $w_{4}$ | $w_{5}$ | $w_{6}$ | $w_{7}$ |
| Hesitant fuzzy <br> programming <br> method | 0.1346 | 0.0646 | 0.0646 | 0.3918 | 0.2256 | 0.0475 | 0.0712 |
| Eigenvector <br> method | 0.1302 | 0.0524 | 0.0524 | 0.3798 | 0.2669 | 0.0343 | 0.0843 |

In Table 3.11, the priorities of the projects are slightly different, each project gets a priority which represents the relative importance of the project. To compare the two methods, we define a solution preference relation, $W=\left(w_{i j}\right)_{n \times n}$, where $w_{i j}=\frac{w_{i}}{w_{j}}$ are the ratios of the priorities. Assume a multiplicative preference relation $B=\left(b_{i j}\right)_{n \times n}$, the Euclidean distance (Golany and Kress 1993) is utilized to measure the deviation degree between $W$ and $B$, defined as follows:

$$
\begin{equation*}
d(W, B)=\sqrt{\frac{2}{n(n-1)}\left(\sum_{i, j=1 ; i<j}^{n}\left(w_{i j}-b_{i j}\right)^{2}\right)} \tag{3.110}
\end{equation*}
$$

The original crisp pairwise comparison judgments of the water conservancy projects shown in Table 3.11 can be constructed as a multiplicative preference relation $B^{\text {proj. }}=\left(b_{i j}^{\text {proj. }}\right)_{7 \times 7}$. According to Table 3.11, let $W^{E V M}=\left(w_{i j}^{E V M}\right)_{7 \times 7}$ and $W^{H F P M}=\left(w_{i j}^{H F P M}\right)_{7 \times 7}$ be the solution preference relations constructed by the ratios of priorities derived by the eigenvector method and the hesitant fuzzy programming method, respectively. By Eq.(3.110), we have $d\left(W^{E V M}, B^{\text {proj. }}\right)=0.5249$, $d\left(W^{\text {HFPM }}, B^{\text {proj. }}\right)=0.4841$. Obviously, $d\left(W^{H F P M}, B^{\text {proj. }}\right)<d\left(W^{E V M}, B^{\text {proj. }}\right)$, it means that the hesitant fuzzy programming method gives a better approximation to the original preference relation $B^{\text {proj. }}$. Furthermore, according to Table 3.11, the ranking results of the seven water conservancy projects are the same by the two optimization methods, that is, $w_{4}>w_{5}>w_{1}>w_{7}>w_{2}=w_{3}>w_{6}$. Therefore, the hesitant fuzzy programming method can be considered as an alternative of the eigenvector method, and it performs better than the eigenvector method.

As Zhang (2009) introduced, twenty-seven DMs (experts) are required to provide full pairwise comparison judgments at each level of hierarchy, and they should necessarily agree upon all the comparison judgments at each level so as to obtain crisp judgments. Some prioritization procedures should be used for the group to agree upon the judgments, such as consensus, voting, or aggregated methods Zhang (2009). Moreover, in order to get meaningful results by AHP, the consistency degree should be satisfactory at each judgment level. Thus, it appears to be a complex and difficult process to obtain a desired result for so many DMs and complicated hierarchy structure by the traditional AHP-GDM.

Assume that the twenty-seven DMs as a group, do not consider individuals, we can regard this as AHP-HGDM and use the hesitant fuzzy programming method to deal with the optimization problem. Since the twenty-seven DMs act as a group, the comparison judgments from the group can be constructed as a hesitant multiplicative preference relation directly without prioritization procedures. If the group is unwilling or unable to provide some comparison judgments, the hesitant fuzzy programming method can deal with this situation where there are missing judgments. The group also does not need to ensure the consensus degree of their judgments. Therefore, with no necessary prioritization procedures, full comparison judgments, and the satisfied consensus degree, this water conservancy problem is greatly simplified under AHP-HGDM.

According to Zhang (2009)'s results, the total investment ratio of the Southeast Rivers and Southwest Rivers is less than $10 \%$ both in short-term and medium-term (Fig. 3.5 (Zhang 2009)). For simplicity, we concentrate on the other seven major river basins in this paper.


Fig. 3.5. The reasonable allocation of WCICC among river basins in China

The resolution process of this problem is based on the four main steps of AHP-HGDM introduced in Subsection 3.6.1. Therefore, we give the following four steps for obtaining the reasonable allocation of WCICC among the seven major river basins in China (Zhu and Xu 2013b).

Step 1. Let the reasonable allocation be the goal level, the seven kinds of projects as the criterion level and the seven major river basins as the scheme level, we have the hierarchy structure shown in Fig. 3.6 (Zhu and Xu 2013b):


Fig. 3.6. The hierarchy structure
Step 2. In order to compare with Zhang (2009)'s results, we retain original crisp pairwise comparison judgments with respect to each level of hierarchy, then add some other possible values to original judgments or omit some crisp judgments. These modified comparison judgments can be constructed as hesitant multiplicative preference relations or incomplete hesitant multiplicative preference relations, shown in Tables 3.12 and 3.13 (Zhu and Xu 2013b).

Table 3.12. The modified pairwise comparison judgments of the water conservancy projects

|  | Reservoir <br> project | Irrigation <br> project | Waterlog <br> control <br> project | Flood <br> control and <br> prevention <br> project | Water <br> supply <br> project | hydropower <br> project |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| water and <br> soil <br> conservation <br> project |  |  |  |  |  |  |
| Reservoir <br> project | $\{1\}$ | $\{2,3\}$ | $\{3\}$ | $\left\{\frac{1}{5}, \frac{1}{4}\right\}$ | $\left\{\frac{1}{3}\right\}$ | $\{4\}$ |
| Irrigation <br> project | $\{1\}$ | $\{1\}$ | $\left\{\frac{1}{6}\right\}$ | $\left\{\frac{1}{5}\right\}$ | $\{2\}$ | $\left\{\frac{1}{2}\right\}$ |
| Waterlog <br> control <br> project |  | $\{1\}$ | $\left\{\frac{1}{7}, \frac{1}{6}\right\}$ | $\{1 / 5\}$ | $\{1,2\}$ | - |
| Flood control <br> and <br> prevention <br> project |  |  |  | $\{1\}$ | $\{2\}$ | $\{7,8\}$ |
| Water supply <br> project |  |  |  |  | $\{5\}$ |  |
| hydropower <br> project |  |  |  |  |  | $\{5,6\}$ |
| water and soil <br> conservation <br> project |  |  |  |  |  | $\left\{\frac{1}{3}\right\}$ |

Table 3.13. The modified pairwise comparison judgments of the seven river basins with respect to certain project

| Yangtse | Yellow | Haihe and | Song and | Inland | Huaihe | Zhujiang |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| River | River | Luanhe | Liao Rivers | Rivers | River | River |
|  |  | Rivers |  |  |  |  |

Reservoir project


| Yangtse River $\quad\{1\}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ | $\left\{\frac{1}{3}\right\}$ | $\left\{1, \frac{1}{2}\right\}$ | $\left\{\frac{1}{4}\right\}$ | $\left\{\frac{1}{3}\right\}$ | $\{3,4\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Yellow River | \{1\} | \{3\} | \{4\} | \{2\} | \{3\} | $\{7,8\}$ |
| Haihe and Luanhe Rivers |  | \{1\} | \{2\} | $\left\{\frac{1}{2}\right\}$ | \{1\} | \{5\} |
| Song and Liao Rivers |  |  | \{1\} | - | $\left\{\frac{1}{2}\right\}$ | - |
| Inland Rivers |  |  |  | \{1\} | \{2\} | \{6\} |
| Huaihe River |  |  |  |  | \{1\} | \{5\} |
| Zhujiang River |  |  |  |  |  | \{1\} |
| Waterlog contro1 project |  |  |  |  |  |  |
| Yangtse River $\{1\}$ | \{7\} | \{7\} | $\{3,4\}$ | \{9\} | - | \{3\} |
| Yellow River | \{1\} | \{1\} | - | $\{2,3\}$ | $\left\{\frac{1}{6}, \frac{1}{5}\right\}$ | $\left\{\frac{1}{5}\right\}$ |
| Haihe and Luanhe Rivers |  | \{1\} | $\left\{\frac{1}{5}\right\}$ | \{3\} | $\left\{\frac{1}{6}\right\}$ | $\left\{\frac{1}{5}\right\}$ |
| Song and Liao Rivers |  |  | \{1\} | \{7\} | $\left\{\frac{1}{2}\right\}$ | - |
| Inland Rivers |  |  |  | \{1\} | - | $\left\{\frac{1}{7}, \frac{1}{8}\right\}$ |
| Huaihe River |  |  |  |  | \{1\} | $\{2,3\}$ |
| Zhujiang River |  |  |  |  |  | \{1\} |

Table 3.13. (continued)


Table 3.13. (continued)

| Water and soil conservation project |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Yangtse River <br> $\{1\} \quad\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ | $\{2,3,4\}$ | $\{4,5\}$ | $\{3,4\}$ | $\{4,5\}$ | $\{5,6,7\}$ |
| Yellow River $\quad\{1\}$ | \{3\} | \{6\} | \{5\} | \{5\} | \{6\} |
| Haihe and Luanhe Rivers | \{1\} | \{4\} | $\left\{\frac{1}{2}, 3\right\}$ | \{3\} | $\left\{\frac{1}{3}, \frac{1}{4}\right\}$ |
| Song and Liao Rivers |  | \{1\} | $\left\{\frac{1}{2}\right\}$ | $\left\{\frac{1}{2}\right\}$ | \{1\} |
| Inland Rivers |  |  | \{1\} | \{1\} | \{2\} |
| Huaihe River |  |  |  | \{1\} | $\{2\}$ |
| Zhujiang River |  |  |  |  | \{1\} |

Step 3. By the hesitant fuzzy programming method, the final group solution of the seven projects, and the final group solutions of the seven major river basins with respect to the certain project can be obtained shown in Tables 3.14 and 3.15 (Zhu and Xu 2013b), respectively.

Table 3.14. The final group solution of the seven projects

| $G C I$ | Reservoir <br> project | Irrigation <br> project | Waterlog <br> control <br> project | Flood control <br> and prevention <br> project | Water <br> supply <br> project | Hydro <br> power <br> project | water and soil <br> conservation <br> project |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.9513 | 0.1217 | 0.0852 | 0.0568 | 0.3895 | 0.2191 | 0.0426 | 0.0852 |

Table 3.15. The final group solutions of the major seven river basins with respect to certain project

|  | Reservoir <br> project | Irrigation <br> project | Waterlog <br> control <br> project | Flood <br> control and <br> prevention <br> project | Water <br> supply <br> project | Hydro <br> power <br> project | water and <br> soil <br> conservation <br> project |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $G C I$ | 0.9381 | 0.9472 | 0.9393 | 0.9261 | 0.9485 | 0.9070 | 0.9384 |
| Yangtse River | 0.4487 | 0.0938 | 0.2883 | 0.3399 | 0.0815 | 0.4590 | 0.2466 |
| Yellow River | 0.0851 | 0.3343 | 0.0499 | 0.1034 | 0.2060 | 0.0920 | 0.3699 |
| Haihe and <br> Luanhe Rivers | 0.0638 | 0.1290 | 0.0499 | 0.0443 | 0.3090 | 0.0462 | 0.1438 |
| Song and Liao <br> Rivers | 0.1702 | 0.0909 | 0.1163 | 0.2069 | 0.1202 | 0.1104 | 0.0514 |
| Inland Rivers | 0.0774 | 0.1935 | 0.0253 | 0.0296 | 0.1202 | 0.0407 | 0.0685 |
| Huaihe River | 0.0774 | 0.1232 | 0.3540 | 0.1379 | 0.1202 | 0.0678 | 0.0685 |
| Zhujiang River | 0.0774 | 0.0352 | 0.1163 | 0.1379 | 0.0429 | 0.1840 | 0.0514 |

Step 4. In the last step, the priorities of the seven major river basins are calculated, then the reasonable allocation of WCICC among different projects can be determined. We normalize the priorities obtained by the eigenvector method for the seven major river basins given by Zhang (2009), we can compare the results shown in Table 3.16.

Table 3.16. The priorities of the major seven river basins

|  | Yangtse <br> River | Yellow <br> River | Haihe <br> and <br> Luanhe <br> Rivers | Song <br> and <br> Liao <br> Rivers | Inland <br> Rivers | Huaihe <br> River | Zhujiang <br> River |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Priorities | $w_{1}$ | $w_{2}$ | $w_{3}$ | $w_{4}$ | $w_{5}$ | $w_{6}$ | $w_{7}$ |
| Hesitant fuzzy <br> programming <br> method | 0.2698 | 0.1625 | 0.1208 | 0.1511 | 0.0728 | 0.1288 | 0.0944 |
| Eigenvector <br> method | 0.2475 | 0.1550 | 0.1276 | 0.1474 | 0.0775 | 0.1389 | 0.1060 |

From Table 3.16, we can see that the ranking results of the seven river basins according to the priorities are the same, that is, Yangtse River $\succ$ Yellow River $\succ$ Song and Liao Rivers $\succ$ Huaihe River $\succ$ Haihe and Luanhe Rivers $\succ$ Zhujiang River $\succ$ Inland Rivers. In the total WCICC, Yangtse River basin should be invested more than other rivers basins, its priorities are $26.98 \%$ and $24.75 \%$ obtained by the hesitant fuzzy programming method and the eigenvector method, respectively, and the least investment should be allocated to the Inland Rivers. Although the ranking results of the river basins by the two optimization methods are the same, the priorities obtained by the eigenvector method reflect a compromise solution, while the hesitant fuzzy programming method results in a best solution. To compare the two solutions, we can use the standard deviation of priorities, since a smaller value of the standard deviation of priorities, a more compromise solution it should be. According to Table 3.16, we let

$$
\begin{aligned}
& w^{H F P M}=(0.2698,0.1625,0.1208,0.151,0.0728,0.1288,0.0944)^{\mathrm{T}} \\
& w^{E V M}=(0.2475,0.1550,0.1276,0.1474,0.0775,0.1389,0.1060)^{\mathrm{T}}
\end{aligned}
$$

be the two solutions of priority vector obtained by the hesitant fuzzy programming method and the eigenvector method, respectively. Then their standard deviations are

$$
\begin{gathered}
\sigma_{H F P M}=\sqrt{\frac{1}{7} \sum_{i=1}^{7}\left(w_{i}^{H F P M}-\bar{w}^{H F P M}\right)^{2}}=0.0592 \\
\sigma_{E V M}=\sqrt{\frac{1}{7} \sum_{i=1}^{7}\left(w_{i}^{E V M}-\bar{w}^{E V M}\right)^{2}}=0.0493
\end{gathered}
$$

where $\bar{w}^{H F P M}$ and $\bar{w}^{E V M}$ are the means of $w^{H F P M}$ and $w^{E V M}$, respectively. Obviously, we have $\sigma_{E V M}<\sigma_{H F P M}$, the solution obtained by the hesitant fuzzy programming method is a better one.

The illustrative example of water conservancy in China with the comparison analysis between the hesitant fuzzy programming method and the eigenvector method does not pretend to be a comprehensive one. The main objective of this example is to show the application of AHP-HGDM in practice and the hesitant fuzzy programming method with desirable features. The preliminary results in this paper confirm that as an extension of the AHP-GDM, AHP-HGDM performs better than traditional methods. It can be considered as a new tool to deal with group decision making, and the hesitant fuzzy programming method is an effective and convenient prioritization method, which makes the resolution process of AHP-HGDM much easier.

## Chapter 4 <br> Hesitant Fuzzy MADM Models

Multi-attribute decision making (MADM), which addresses the problem of making an optimal choice that has the highest degree of satisfaction from a set of alternatives that are characterized in terms of their attributes, is a usual task in human activities. In classical MADM, the assessments of alternatives are precisely known (Dyer et al. 1992; Stewart 1992). However, because of the inherent vagueness of human preferences as well as the objects being fuzzy and uncertain, the attributes involved in decision making problems are not always expressed in real numbers, and some are better suited to be denoted by fuzzy values, such as interval values (Cao and Wu 2011; Yue 2011; Zhang and Liu 2010; Xu 2004b, 2005c; Xu and Chen 2008d,e; Xu and Da 2004), fuzzy numbers (Xu and Chen 2007), linguistic variables (Fan and Feng 2009; Parreiras et al. 2010; Yu et al. 2012; Xu 2004c, 2007d,e, 2009b,c), intuitionistic fuzzy numbers (IFNs) (Xu and Yager 2006, 2008; Xu and Cai 2010b; Xu 2012), and hesitant fuzzy elements (HFEs) (Xu and Xia 2011b,c; Xia and Xu 2011a; Zhu et al. 2012a), just to mention a few. Since Bellman and Zadeh (1970) first proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics in 1970, fuzzy MADM has been receiving more and more attention. Many methods for MADM, such as the TOPSIS method (Technique for Order Preference by Similarity to Ideal Solution) (Jahanshahloo et al. 2006; Wang and Elhag 2006), the maximizing deviation method (Xu 2005d, 2010c; Wu and Chen 2007), the VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje) method (Opricovic and Tzeng 2004, 2007), the PROMETHEE (Preference Ranking Organisation METHod for Enrichment Evaluations) method (Brans 1984), and the ELECTRE (ELimination Et Choix Traduisant la REalité) method (Roy 1996) have been extended to take different types of attribute values into account, such as interval values, linguistic variables, and IFNs. All of the above methods, however, have not yet been accommodated to fit the hesitant fuzzy assessments provided by the DMs. HFEs (Xu and Xia 2011b,c) (which can be seen as the basic elements of HFSs (Torra and Narukawa 2009, 2010) describes the situations that permit the membership of an element to a given set having a few different values, which is a useful means to describe and deal with uncertain information in the process of MADM.

Xia and Xu (2011a) developed some aggregation operators for hesitant fuzzy information, and gave their application for solving the MADM problems under hesitant fuzzy environment. Xu and Xia (2011b) gave a detailed study on distance
and similarity measures for HFSs and proposed an approach based on distance measures for the MADM problems. Xia et al. (2013a) also proposed some other hesitant fuzzy aggregation techniques and applied them in group decision making. Yu et al. (2011) proposed a hesitant fuzzy Choquet integral operator and applied it in MADM under hesitant fuzzy environment in which the weight vector of attributes is exactly known. Wei (2012) also developed some prioritized aggregation operators for hesitant fuzzy information, and developed some models for hesitant fuzzy MADM problems in which the attributes are in different priority levels. Yu et al. (2012) proposed the generalized hesitant fuzzy Bonferroni mean to solve the MAGDM problems where the attributes are correlative under hesitant fuzzy environment. More recently, Qian et al. (2013) generalized the HFSs using intuitionistic fuzzy sets in group decision making framework. The generalized HFS is fit for the situations when the DMs have a hesitation among several possible memberships under uncertainty. Chen et al. (2013b) also generalized the concept of HFS to that of interval-valued hesitant fuzzy set (IVHFS) in which the membership degrees of an element to a given set are not exactly defined, but denoted by several possible interval values, and meanwhile developed an approach to group decision making based on interval-valued hesitant preference relations in order to consider the differences of opinions between individual DMs. Obviously, most of these papers put their emphasis on the extensions of the aggregation techniques in MADM under hesitant fuzzy scenarios. However, when using these techniques, the associated weighting vector is more or less determined subjectively and the decision information itself is not taken into consideration sufficiently; More importantly, a significant pitfall of the aforementioned methods is the need for the information about attribute weights being exactly known. To solve this issues, Xu and Zhang (2013) developed an approach based on the TOPSIS method and the maximizing deviation method for solving the MADM problems, in which the evaluation information provided by the DM is expressed in HFEs and the information about attribute weights is incomplete. There are two key issues being addressed in this approach. The first one is to establish an optimization model based on the maximizing deviation method, which can be used to determine the attribute weights. According to the idea of the TOPSIS method of Hwang and Yoon (1981), the second one is to calculate the relative closeness coefficient of each alternative to the hesitant positive-ideal solution, based on which the considered alternatives are ranked and then the most desirable one is selected. Chen et al. (2013c) developed a hesitant fuzzy ELECTRE I method and applied it to solve the MADM problem under hesitant fuzzy environments. The new method is formulated using the concepts of hesitant fuzzy concordance and hesitant fuzzy discordance which are based on the given scores and the deviation degrees, and employed to determine the preferable alternative. Motivated by the TOPSIS method, Liao and Xu (2013a) defined the satisfaction degree of the alternative, based on which several optimization models are derived to determinate the weights of attributes, and then they developed an interactive method based on some optimization models for the MADM problems with hesitant fuzzy information.

### 4.1 Hesitant Fuzzy MADM Based on TOPSIS with Incomplete Weight Information

This section puts forward a framework for determining the attribute weights and the ranking orders for all the alternatives with incomplete weight information under hesitant fuzzy environment.

A MADM problem can be expressed as a decision matrix whose elements indicate the evaluation information of all alternatives with respect to an attribute. We construct a hesitant fuzzy decision matrix, whose elements are HFEs, which are given due to the fact that the membership degree of the considered alternative satisfying a given attribute may originate from a doubt between a few different values.

Consider a MADM problem where there is a discrete set of $n$ alternatives, $A=\left\{A_{1}, A_{2}, \ldots, A_{n}\right\}$. Let $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ be the discussion universe containing the attributes. A HFS of the $i$ th alternative $A_{i}$ on $X$ is given by Xia and Xu (2011a):

$$
\begin{equation*}
A_{i}=\left\{<x_{j}, h_{A_{i}}\left(x_{j}\right)>\mid x_{j} \in X\right\} \tag{4.1}
\end{equation*}
$$

where $h_{A_{i}}\left(x_{j}\right)=\left\{\gamma \mid \gamma \in h_{A_{i}}\left(x_{j}\right), 0 \leq \gamma \leq 1\right\}, \quad i=1,2, \cdots m ; j=1,2, \cdots, n . h_{Y_{i}}\left(x_{j}\right)$ indicates the possible membership degrees of the $i$ th alternative $A_{i}$ under the $j$ th attribute $x_{j}$, and it can be expressed as a HFE $h_{i j}$. The hesitant fuzzy decision matrix $H$, can be written as:

$$
H=\left[\begin{array}{cccc}
h_{11} & h_{12} & \cdots & h_{1 n}  \tag{4.2}\\
h_{21} & h_{22} & \cdots & h_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
h_{m 1} & h_{m 2} & \cdots & h_{m n}
\end{array}\right]
$$

Considering that the attributes have different importance degrees, the weight vector of all the attributes, given by the DMs, is defined by $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$, where $0 \leq w_{j} \leq 1, j=1,2, \ldots, n, \sum_{j=1}^{n} w_{j}=1$, and $w_{j}$ is the importance degree of each attribute. Due to the complexity and uncertainty of practical decision making problems and the inherent subjective nature of human thinking, the information about attribute weights is usually incomplete. For
convenience, let $\Delta$ be a set of the known weight information (Park and Kim 1997; Kim and Ahn 1999; Kim and Han 1999; Park 2004; Xu and Chen 2007; Xu and Xia 2012b; Xu 2007d,e), where $\Delta$ can be constructed by the following forms, for $i \neq j$ :

Form 1. A weak ranking: $\left\{w_{i} \geq w_{j}\right\}$.
Form 2. A strict ranking: $\left\{w_{i}-w_{j} \geq \alpha_{i}\right\}\left(\alpha_{i}>0\right)$.
Form 3. A ranking of differences: $\left\{w_{i}-w_{j} \geq w_{k}-w_{l}\right\}$, for $j \neq k \neq l$.

Form 4. A ranking with multiples: $\left\{w_{i} \geq \alpha_{i} w_{j}\right\}\left(0 \leq \alpha_{i} \leq 1\right)$.

Form 5. An interval form: $\left\{\alpha_{i} \leq w_{i} \leq \alpha_{i}+\varepsilon_{i}\right\}\left(0 \leq \alpha_{i} \leq \alpha_{i}+\varepsilon_{i} \leq 1\right)$.

The estimation of the attribute weights plays an important role in MADM. The maximizing deviation method was proposed by Wang (1998) to determine the attribute weights for solving the MADM problems with numerical information. According to Wang (1998), for a MADM problem, the attribute with a larger deviation value among alternatives should be assigned a larger weight, while the attribute with a small deviation value among alternatives should be signed a smaller weight. In other word, if the performance values of all the alternatives have small differences under an attribute, it shows that such an attribute plays a less important role in the priority procedure. On the contrary, if an attribute makes the performance values of all the alternatives have obvious differences, then this attribute plays a much important role in choosing the best alternative. So from the standpoint of ranking the alternatives, if one attribute has similar attribute values across alternatives, it should be assigned a small weight; Otherwise, the attribute which makes larger deviations should be evaluated a bigger weight, in spite of the degree of its own importance. Especially, if all available alternatives score equally with respect to a given attribute, then such an attribute will be judged unimportant by most of DMs. Wang (1998) suggested that zero weight should be assigned to the corresponding attribute.

Xu and Zhang (2013) constructed an optimization model based on the maximizing deviation method to determine the optimal relative weights of attributes under hesitant fuzzy environment.

For the attribute $x_{j} \in X$, the deviation of the alternative $A_{i}$ to all the other alternatives can be expressed as:

$$
\begin{equation*}
d_{i j}(w)=\sum_{k=1}^{n} d\left(h_{i j}, h_{k j}\right) w_{j}, i=1,2, \cdots, n ; j=1,2, \cdots, m \tag{4.3}
\end{equation*}
$$

where $d\left(h_{i j}, h_{k j}\right)=\sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}{ }^{\sigma(q)}-h_{k j}{ }^{\sigma(q)}\right|^{2}}$ denotes the hesitant Euclidean distance between the HFEs $h_{i j}$ and $h_{k j}$ defined as in Section 2.2.

Let

$$
\begin{equation*}
d_{j}(w)=\sum_{i=1}^{n} d_{i j}(w)=\sum_{i=1}^{n} \sum_{k=1}^{n} w_{j} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}}, j=1,2, \cdots, m \tag{4.4}
\end{equation*}
$$

then $d_{j}(w)$ represents the deviation value of all alternatives to other alternatives for the attribute $x_{j} \in X$.

Based on the above analysis, we can construct a non-linear programming model to select the weight vector $w$ which maximizes all deviation values for all the attributes, as follows ( Xu and Zhang 2013):
(M-4.1)

$$
\begin{cases}\max & d(w)=\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} w_{j} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}} \\ \text { s.t. } \quad w_{j} \geq 0, \quad j=1,2, \cdots, m, \sum_{j=1}^{m} w_{j}^{2}=1 .\end{cases}
$$

To solve the above model, we let

$$
\begin{equation*}
f(w, \boldsymbol{\xi})=\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{\lambda=1}^{l}\left|h_{i j}{ }^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}} w_{j}+\frac{\xi}{2}\left(\sum_{j=1}^{m} w_{j}{ }^{2}-1\right) \tag{4.5}
\end{equation*}
$$

which indicates the Lagrange function of the constrained optimization problem (M-4.1), where $\xi$ is a real number, denoting the Lagrange multiplier variable. Then the partial derivatives of $f$ are computed as:

$$
\begin{equation*}
\frac{\partial f}{\partial w_{j}}=\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}}+\xi w_{j}=0 \tag{4.6}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\partial f}{\partial \xi}=\frac{1}{2}\left(\sum_{j=1}^{m} w_{j}^{2}-1\right)=0 \tag{4.7}
\end{equation*}
$$

It follows from Eq.(4.6) that

$$
\begin{equation*}
w_{j}=\frac{-\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}}}{\xi}, j=1,2, \cdots, m \tag{4.8}
\end{equation*}
$$

Putting Eq.(4.8) into Eq.(4.7), we have

$$
\begin{equation*}
\xi=-\sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}}\right)^{2}} \tag{4.9}
\end{equation*}
$$

Obviously, $\quad \xi<0, \quad \sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}{ }^{\sigma(q)}\right|^{2}}$ means the sum of deviations of all the alternatives with respect to the $j$ th attribute, and $\sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l} h_{j}^{\sigma(q)}-\left.h_{k j}^{\sigma(q)}\right|^{2}}\right)^{2}}$ means the sum of deviations of all the alternatives with respect to all the attributes.

Then combining Eqs.(4.8) and (4.9), we can get

$$
\begin{equation*}
w_{j}=\frac{\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{\lambda=1}^{l}\left|h_{i j}{ }^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}}}{\sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}{ }^{\sigma(q)}-h_{k j}{ }^{\sigma(q)}\right|^{2}}\right)^{2}}} \tag{4.10}
\end{equation*}
$$

For the sake of simplicity, let $d_{j}=\sum_{i=1}^{n} \sum_{k=1}^{n} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}{ }^{\sigma(q)}\right|^{2}}$, $j=1,2, \cdots, m$, and Eq.(4.10) can be rewritten as:

$$
\begin{equation*}
w_{j}=\frac{d_{j}}{\sqrt{\sum_{j=1}^{m} d_{j}^{2}}}, j=1,2, \cdots, m \tag{4.11}
\end{equation*}
$$

It can be verified from Eq.(4.11) easily that $w_{j}(j=1,2, \cdots, m)$ are positive such that they satisfy the constrained conditions in the model (M-4.1) and the solution is unique.

By normalizing $w_{j}(j=1,2, \cdots, n)$, we make their sum into a unit, and get

$$
\begin{equation*}
w_{j}^{*}=\frac{w_{j}}{\sum_{j=1}^{n} w_{j}}, j=1,2, \cdots, m \tag{4.12}
\end{equation*}
$$

However, there are actual situations that the information about the weight vector is not completely unknown but partially known. For these cases, based on the set of the known weight information, $\Delta$, we construct the following constrained optimization model (Xu and Zhang 2013):
(M-4.2)

$$
\left\{\begin{array}{l}
\max \quad d(w)=\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} w_{j} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-h_{k j}^{\sigma(q)}\right|^{2}} \\
\text { s.t } \quad w \in \Delta, \quad w_{j} \geq 0, \quad j=1,2, \cdots, m, \sum_{j=1}^{m} w_{j}=1
\end{array}\right.
$$

where $\Delta$ is also a set of constraint conditions that the weight value $w_{j}$ should satisfy according to the requirements in real situations.

The model (M-4.2) is a linear programming model that can be executed using the MATLAB 7.4.0 mathematics software package. By solving this model, we get the optimal solution $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$, which can be used as the weight vector of attributes.

In general, after obtaining the attribute weight values on basis of the maximizing deviation method, analogous to the literature ( Xu 2010 c ; Wu and Chen 2007), we should utilize a certain kind of operator to aggregate the given decision information so as to get the overall preference value of each alternative, and then rank the alternatives and select the most desirable one(s). In the process of hesitant fuzzy information aggregation, however, it produces the loss of too much information due to the complexity of the aggregation process of hesitant fuzzy aggregation operators, which implies a lack of precision in the final results. Therefore, in order to overcome this disadvantage, Xu and Zhang (2013) extended the TOPSIS method to take hesitant fuzzy information into account and utilized the distance measures of

HFEs to obtain the final ranking of the alternatives. TOPSIS, proposed by Hwang and Yoon (1981), is a kind of method to solve the MADM problems, which aims at choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS), and is widely used for tackling the ranking problems in real situations.

Under hesitant fuzzy environment, the hesitant fuzzy PIS, denoted by $A^{+}$, and the hesitant fuzzy NIS, denoted by $A^{-}$, can be defined as follows:

$$
\begin{align*}
A^{+}= & \left\{\left\langle x_{j}, \max _{i}\left\{h_{i j}^{\sigma(q)}\right\}\right\rangle \mid j=1,2, \cdots, m\right\} \\
= & \left\{\left\langle x_{1},\left\{\left(h_{1}^{1}\right)^{+},\left(h_{1}^{2}\right)^{+}, \cdots,\left(h_{1}^{l}\right)^{+}\right\}\right\rangle,\left\langle x_{2},\left\{\left(h_{2}^{1}\right)^{+},\left(h_{2}^{2}\right)^{+}, \cdots,\left(h_{2}^{l}\right)^{+}\right\}\right\rangle, \cdots,\right. \\
& \left.\left\langle x_{m},\left\{\left(h_{m}^{1}\right)^{+},\left(h_{m}^{2}\right)^{+}, \cdots,\left(h_{m}^{l}\right)^{+}\right\}\right\rangle\right\}  \tag{4.13}\\
A^{-}= & \left\{\left\langle x_{j}, \min _{i}\left\{h_{i j}^{\sigma(q)}\right\}\right\rangle \mid j=1,2, \cdots, m\right\} \\
= & \left\{\left\langle x_{1},\left\{\left(h_{1}^{1}\right)^{-},\left(h_{1}^{2}\right)^{-}, \cdots,\left(h_{1}^{l}\right)^{-}\right\}\right\rangle,\left\langle x_{2},\left\{\left(h_{2}^{1}\right)^{-},\left(h_{2}^{2}\right)^{-}, \cdots,\left(h_{2}^{l}\right)^{-}\right\}\right\rangle, \cdots,\right. \\
& \left.\left\langle x_{m},\left\{\left(h_{m}^{1}\right)^{-},\left(h_{m}^{2}\right)^{-}, \cdots,\left(h_{m}^{l}\right)^{--}\right\}\right\rangle\right\} \tag{4.14}
\end{align*}
$$

The separation between alternatives can be measured by Hamming distance or Euclidean distance. In order to measure the distances between HFEs, we adopt the hesitant fuzzy Euclidean distance proposed by Xu and Xia (2011c). The separation measures, $d_{i}^{+}$and $d_{i}^{-}$, of each alternative from the hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$, respectively, are derived from

$$
\begin{gather*}
d_{i}^{+}=\sum_{j=1}^{m} d\left(h_{i j}, h_{j}^{+}\right) w_{j}=\sum_{j=1}^{m} w_{j} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-\left(h_{j}^{\sigma(q)}\right)^{+}\right|^{2}}, i=1,2, \cdots, n  \tag{4.15}\\
d_{i}^{-}=\sum_{j=1}^{m} d\left(h_{i j}, h_{j}^{-}\right) w_{j}=\sum_{j=1}^{m} w_{j} \sqrt{\frac{1}{l} \sum_{q=1}^{l}\left|h_{i j}^{\sigma(q)}-\left(h_{j}^{\sigma(q)}\right)^{-}\right|^{2}}, i=1,2, \cdots, n \tag{4.16}
\end{gather*}
$$

The relative closeness coefficient of an alternative $A_{i}$ with respect to the hesitant fuzzy PIS $A^{+}$is defined as the following formula:

$$
\begin{equation*}
c\left(A_{i}\right)=\frac{d_{i}^{-}}{d_{i}^{+}+d_{i}^{-}}, \quad i=1,2, \cdots, n \tag{4.17}
\end{equation*}
$$

where $0 \leq c\left(A_{i}\right) \leq 1, i=1,2, \cdots, n$. Obviously, an alternative $A_{i}$ is closer to the hesitant fuzzy PIS $\left(A^{+}\right)$and farther from the hesitant fuzzy NIS $\left(A^{-}\right)$as $c\left(A_{i}\right)$ approaches 1 . Therefore, according to the closeness coefficient $c\left(A_{i}\right)$, we can determine the ranking orders of all alternatives and select the best one from a set of feasible alternatives.

Based on the above models, Xu and Zhang (2013) developed a practical approach for solving the MADM problems, in which the information about attribute weights is incompletely known or completely unknown, and the attribute values take the form of hesitant fuzzy information. The schematic diagram of the proposed approach for MADM is provided in Fig. 4.1 (Xu and Zhang 2013). The approach involves the following steps:
Step 1. For a MADM problem, we construct the decision matrix $H=\left(h_{i j}\right)_{n \times m}$, where all the arguments $h_{i j}(i=1,2, \ldots, n ; j=1,2, \ldots, m)$ are HFEs, given by the DMs , for the alternative $A_{i} \in A$ with respect to the attribute $x_{j} \in X$.

Step 2. If the information about the attribute weights is completely unknown, then we can obtain the attribute weights by using Eq.(4.12); If the information about the attribute weights is partly known, then we solve the model (M-4.2) to obtain the attribute weights.
Step 3. Utilize Eqs.(4.13) and (4.14) to determine the corresponding hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$.

Step 4. Utilize Eqs.(4.15) and (4.16) to calculate the separation measures $d_{i}^{+}$and $d_{i}^{-}$of each alternative $A_{i}$ from the hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$, respectively.

Step 5. Utilize Eq.(4.17) to calculate the relative closeness coefficient $c\left(A_{i}\right)$ of each alternative $A_{i}$ to the hesitant fuzzy PIS $A^{+}$.

Step 6. Rank the alternatives $A_{i}(i=1,2, \cdots, n)$ according to the relative closeness coefficients $c\left(A_{i}\right)(i=1,2, \cdots, n)$ to the hesitant fuzzy PIS $A^{+}$and then select the most desirable one(s).


Fig. 4.1. The schematic diagram of the proposed approach for MADM

In the following, we use an energy police selection problem (adopted from Xu and Xia (2011b); Kahraman and Kaya (2010)) to demonstrate the applicability and the implementation process of our approach under hesitant fuzzy environment.

Example 4.1 ( Xu and Zhang 2013). Energy is an indispensable factor for the social and economic development of societies. The correct energy policy affects economic development and environment, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) $A_{i}(i=1,2,3,4,5)$, and four attributes: $x_{1}$ : Technological; $x_{2}$ : Environmental; $x_{3}$ : Socio-political; $x_{4}$ : Economic. Several DMs are invited to evaluate the performances of the five alternatives. For an alternative under an attribute, although all of the DMs provide their evaluation values, some of these values may be repeated. However, a value repeated more times does not indicate that it has more importance than other values repeated less times. To get a more reasonable result, it is better that the DMs give their evaluations anonymously. We only collect all of the possible values for an alternative under an attribute, and each value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once. The HFE is just a tool to deal with such cases, and all possible evaluations for an alternative under the attributes can be considered as a HFE. The results evaluated by the DMs are contained in a hesitant fuzzy decision matrix, shown in Table 4.1 (Xu and Zhang 2013).

Table 4.1. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5,0.4,0.3\}$ | $\{0.9,0.8,0.7,0.1\}$ | $\{0.5,0.4,0.2\}$ | $\{0.9,0.6,0.5,0.3\}$ |
| $A_{2}$ | $\{0.5,0.3\}$ | $\{0.9,0.7,0.6,0.5,0.2\}$ | $\{0.8,0.6,0.5,0.1\}$ | $\{0.7,0.4,0.3\}$ |
| $A_{3}$ | $\{0.7,0.6\}$ | $\{0.9,0.6\}$ | $\{0.7,0.5,0.3\}$ | $\{0.6,0.4\}$ |
| $A_{4}$ | $\{0.8,0.7,0.4,0.3\}$ | $\{0.7,0.4,0.2\}$ | $\{0.8,0.1\}$ | $\{0.9,0.8,0.6\}$ |
|  |  |  |  |  |
| $A_{5}$ | $\{0.9,0.7,0.6,0.3,0.1\}$ | $\{0.8,0.7,0.6,0.4\}$ | $\{0.9,0.8,0.7\}$ | $\{0.9,0.7,0.6,0.3\}$ |

The hierarchical structure of this decision making problem is shown in Fig. 4.2 ( Xu and Zhang 2013).


Fig. 4.2. The energy policy selection hierarchical structure

Obviously, the numbers of values in different HFEs of HFSs are different. In order to more accurately calculate the distance between two HFSs, we should extend the shorter one until both of them have the same length when we compare them. According to the regulations mentioned above, we consider that the DMs are pessimistic in Example 4.1, and change the hesitant fuzzy data by adding the minimal values as listed in Table 4.2 (Xu and Zhang 2013).

Table 4.2. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.5,0.4,0.3,0.3,0.3\}$ | $\{0.9,0.8,0.7,0.1,0.1\}$ | $\{0.5,0.4,0.2,0.2,0.2\}$ | $\{0.9,0.6,0.5,0.3,0.3\}$ |
|  |  |  |  |  |
| $A_{2}$ | $\{0.5,0.3,0.3,0.3,0.3\}$ | $\{0.9,0.7,0.6,0.5,0.2\}$ | $\{0.8,0.6,0.5,0.1,0.1\}$ | $\{0.7,0.4,0.3,0.3,0.3\}$ |
|  |  |  |  |  |
| $A_{3}$ | $\{0.7,0.6,0.6,0.6,0.6\}$ | $\{0.9,0.6,0.6,0.6,0.6\}$ | $\{0.7,0.5,0.3,0.3,0.3\}$ | $\{0.6,0.4,0.4,0.4,0.4\}$ |
|  | $\{0.8,0.7,0.4,0.3,0.3\}$ | $\{0.7,0.4,0.2,0.2,0.2\}$ | $\{0.8,0.1,0.1,0.1,0.1\}$ | $\{0.9,0.8,0.6,0.6,0.6\}$ |
| $A_{4}$ |  |  |  |  |
| $A_{5}$ | $\{0.9,0.7,0.6,0.3,0.1\}$ | $\{0.8,0.7,0.6,0.4,0.4\}$ | $\{0.9,0.8,0.7,0.7,0.7\}$ | $\{0.9,0.7,0.6,0.3,0.3\}$ |

Then, we utilize the approach developed to get the most desirable alternative(s), which involves the following two cases:

Case 1. Assume that the information about the attribute weights is completely unknown, we get the most desirable alternative(s) according to the following steps:

Step 1. Utilize Eq.(4.12) to get the optimal weight vector:

$$
w=(0.2341,0.2474,0.3181,0.2004)^{\mathrm{T}}
$$

Step 2. Utilize Eqs.(4.13) and (4.14) to determine the hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$, respectively:

$$
\begin{aligned}
A^{-}= & \left\{\left\langle x_{1},\{0.5,0.3,0.3,0.3,0.1\}\right\rangle,\left\langle x_{2},\{0.7,0.4,0.2,0.1,0.1\}\right\rangle,\right. \\
& \left.\left\langle x_{3},\{0.5,0.1,0.1,0.1,0.1\}\right\rangle,\left\langle x_{4},\{0.6,0.4,0.3,0.3,0.3\}\right\rangle\right\} \\
A^{+}= & \left\{\left\langle x_{1},\{0.9,0.7,0.6,0.6,0.6\}\right\rangle,\left\langle x_{2},\{0.9,0.8,0.7,0.6,0.6\}\right\rangle,\right. \\
& \left.\left\langle x_{3},\{0.9,0.8,0.7,0.7,0.7\}\right\rangle,\left\langle x_{4},\{0.5,0.1,0.1,0.1,0.1\}\right\rangle\right\}
\end{aligned}
$$

Step 3. Utilize Eqs.(4.15) and (4.16) to calculate the separation measures $d_{i}^{+}$and $d_{i}^{-}$of each alternative $A_{i}$ from the hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$, respectively:
$d_{1}^{+}=0.3555, d_{1}^{-}=0.1976, d_{2}^{+}=0.3277, d_{2}^{-}=0.2384, d_{3}^{+}=0.2418$
$d_{3}^{-}=0.2905, d_{4}^{+}=0.3865 d_{4}^{-}=0.2040, d_{5}^{+}=0.1702, d_{5}^{-}=0.4023$
Step 4. Utilize Eq.(4.17) to calculate the relative closeness $c\left(A_{i}\right)$ of each alternative $A_{i}$ to the hesitant fuzzy PIS $A^{+}$:

$$
\begin{gathered}
c\left(A_{1}\right)=0.3573, c\left(A_{2}\right)=0.4211, c\left(A_{3}\right)=0.5458 \\
c\left(A_{4}\right)=0.3454, c\left(A_{5}\right)=0.7027
\end{gathered}
$$

Step 5. Rank the alternatives $A_{i}(i=1,2,3,4,5)$, according to the relative closeness coefficients $c\left(A_{i}\right)(i=1,2,3,4,5): A_{5} \succ A_{3} \succ A_{2} \succ A_{1} \succ A_{4}$, and thus the most desirable alternative is $A_{5}$.

Case 2. The information about the attribute weights is partly known and the known weight information is given as follows:

$$
\begin{gathered}
\Delta=\left\{w=\left(w_{1}, w_{2}, \ldots, w_{m}\right)^{\mathrm{T}} \mid 0.15 \leq w_{1} \leq 0.2,0.16 \leq w_{2} \leq 0.18,0.3 \leq w_{3} \leq 0.35\right. \\
\left.0.3 \leq w_{4} \leq 0.45, w_{j} \geq 0, j=1,2,3,4, \sum_{j=1}^{4} w_{j}=1\right\}
\end{gathered}
$$

Step 1. Utilize the model (M-4.2) to construct the single-objective model as follows:

$$
\left\{\begin{array}{l}
\max d(w)=4.4467 w_{1}+4.6999 w_{2}+6.0431 w_{3}+3.8068 w_{4} \\
\text { s.t. } \quad w \in \Delta, \quad w_{j} \geq 0, \quad j=1,2,3,4
\end{array}\right.
$$

By solving this model, we get the optimal weight vector:

$$
w=(0.17,0.18,0.35,0.3)^{\mathrm{T}}
$$

Step 2. Utilize Eqs.(4.13) and (4.14) to determine the hesitant fuzzy PIS $A^{+}$and the hesitant fuzzy NIS $A^{-}$, respectively:

$$
\begin{aligned}
A^{-}= & \left\{\left\langle x_{1},\{0.5,0.3,0.3,0.3,0.1\}\right\rangle,\left\langle x_{2},\{0.7,0.4,0.2,0.1,0.1\}\right\rangle,\right. \\
& \left.\left\langle x_{3},\{0.5,0.1,0.1,0.1,0.1\}\right\rangle,\left\langle x_{4},\{0.6,0.4,0.3,0.3,0.3\}\right\rangle\right\} \\
A^{+}= & \left\{\left\langle x_{1},\{0.9,0.7,0.6,0.6,0.6\}\right\rangle,\left\langle x_{2},\{0.9,0.8,0.7,0.6,0.6\}\right\rangle,\right. \\
& \left.\left\langle x_{3},\{0.9,0.8,0.7,0.7,0.7\}\right\rangle,\left\langle x_{4},\{0.9,0.8,0.6,0.6,0.6\}\right\rangle\right\}
\end{aligned}
$$

Step 3. Utilize Eqs.(4.15) and (4.16) to calculate the separation measures $d_{i}^{+}$and $d_{i}^{-}$of each alternative $A_{i}$ :
$d_{1}^{+}=0.3527, d_{1}^{-}=0.1910, d_{2}^{+}=0.3344, d_{2}^{-}=0.2313, d_{3}^{+}=0.2615$
$d_{3}^{-}=0.2645, d_{4}^{+}=0.3865, d_{4}^{-}=0.2200, d_{5}^{+}=0.1641, d_{5}^{-}=0.4088$

Step 4. Utilize Eq.(4.17) to calculate the relative closeness coefficient $c\left(A_{i}\right)$ of each alternative $A_{i}$ to the hesitant fuzzy PIS $A^{+}$:

$$
\begin{gathered}
c\left(A_{1}\right)=0.3514, c\left(A_{2}\right)=0.4089, c\left(A_{3}\right)=0.5027 \\
c\left(A_{4}\right)=0.3653, c\left(A_{5}\right)=0.7136
\end{gathered}
$$

Step 5. Rank the alternatives $A_{i}(i=1,2, \cdots, 5)$, according to the relative closeness coefficient $c\left(A_{i}\right) \quad(\quad i=1,2, \cdots, 5)$. Clearly, $A_{5} \succ A_{3} \succ A_{2} \succ A_{4} \succ A_{1}$, and thus the best alternative is $A_{5}$.

More recently, Nan et al. (2008) proposed a fuzzy TOPSIS method for solving the MADM problems with intuitionistic fuzzy information. Since the HFEs' envelopes are the IFNs, and at same time we consider that under hesitant fuzzy environment, there has no investigation similar to the approach introduced in this section, in the following, we will make a comparison with the intuitionistic fuzzy TOPSIS (IF-TOPSIS) method of Nan et al. (2008), which is the closest to our approach. Considering the HFSs' envelopes, i.e., intuitionistic fuzzy data, and according to Definition 1.6, we can transform the hesitant fuzzy data of the energy police selection problem into the intuitionistic fuzzy data, listed in Table 4.3 (Xu and Zhang 2013). Moreover, since the IF-TOPSIS method needs to know the weight values in advance, hence we also assume the weight vector as $w=(0.2341,0.2474,0.3181,0.2004)^{\mathrm{T}}$.

Table 4.3. Intuitionistic fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $<0.3,0.5>$ | $<0.1,0.1>$ | $<0.2,0.5>$ | $<0.3,0.1>$ |
| $A_{2}$ | $<0.3,0.5>$ | $<0.2,0.1>$ | $<0.1,0.2>$ | $<0.3,0.3>$ |
| $A_{3}$ | $<0.6,0.3>$ | $<0.6,0.1>$ | $<0.3,0.3>$ | $<0.4,0.4>$ |
| $A_{4}$ | $<0.3,0.2>$ | $<0.2,0.3>$ | $<0.1,0.2>$ | $<0.6,0.1>$ |
| $A_{5}$ | $<0.1,0.1>$ | $<0.4,0.2>$ | $<0.7,0.1>$ | $<0.3,0.1>$ |

With the IF-TOPSIS method proposed by Nan et al. (2008), we first need to determine the intuitionistic fuzzy PIS $\tilde{A}^{+}$and the intuitionistic fuzzy NIS $\tilde{A}^{-}$, respectively:

$$
\begin{aligned}
& \tilde{A}^{-}=\left\{\left\langle x_{1},\{0.1,0.1\}\right\rangle,\left\langle x_{2},\{0.1,0.1\}\right\rangle,\left\langle x_{3},\{0.1,0.2\}\right\rangle,\left\langle x_{4},\{0.3,0.1\}\right\rangle\right\} \\
& \tilde{A}^{+}=\left\{\left\langle x_{1},\{0.6,0.3\}\right\rangle,\left\langle x_{2},\{0.6,0.1\}\right\rangle,\left\langle x_{3},\{0.7,0.1\}\right\rangle,\left\langle x_{4},\{0.6,0.1\rangle\right\rangle\right\}
\end{aligned}
$$

Then, according to the distance measure of IFNs:

$$
\begin{equation*}
d\left(A_{i}, A_{k}\right)=\sqrt{\frac{1}{2} \sum_{j=1}^{n} w_{j}\left(\left(\mu_{i j}-\mu_{k j}\right)^{2}+\left(v_{i j}-v_{k j}\right)^{2}+\left(\pi_{i j}-\pi_{k j}\right)^{2}\right)} \tag{4.18}
\end{equation*}
$$

we can calculate the separation measures $\tilde{d}_{i}^{+}$and $\tilde{d}_{i}^{-}$of each alternative $A_{i}$ from the intuitionistic fuzzy PIS $\tilde{A}^{+}$and the intuitionistic fuzzy NIS $\tilde{A}^{-}$, respectively:
$\tilde{d}_{1}^{+}=0.3261, \tilde{d}_{1}^{-}=0.2138, \tilde{d}_{2}^{+}=0.3372, \tilde{d}_{2}^{-}=0.1521, \tilde{d}_{3}^{+}=0.1044$
$\tilde{d}_{3}^{-}=0.4209, \tilde{d}_{4}^{+}=0.3715, \tilde{d}_{4}^{-}=0.1035, \tilde{d}_{5}^{+}=0.2335, \tilde{d}_{5}^{-}=0.2615$

Furthermore, we also calculate the relative closeness coefficient $\tilde{c}\left(A_{i}\right)$ of each alternative $A_{i}$ to the intuitionistic fuzzy PIS $\tilde{A}^{+}$as follows:

$$
\begin{gathered}
\tilde{c}\left(A_{1}\right)=0.3960, \tilde{c}\left(A_{2}\right)=0.3108, \tilde{c}\left(A_{3}\right)=0.7942 \\
\tilde{c}\left(A_{4}\right)=0.2458, \tilde{c}\left(A_{5}\right)=0.5283
\end{gathered}
$$

Finally, according to the relative closeness coefficients $\tilde{c}\left(A_{i}\right)(i=1,2, \cdots, m)$, we rank the alternatives $A_{i}(i=1,2,3,4,5)$ : $A_{3} \succ A_{5} \succ A_{1} \succ A_{2} \succ A_{4}$. Thus the most desirable alternative is $A_{3}$.

It is noticed that the obtained ranking order by Xu and Zhang (2013)'s approach is $A_{5} \succ A_{3} \succ A_{2} \succ A_{1} \succ A_{4}$. Obviously, the ranking order of the alternatives obtained by Nan et al. (2008)'s method is remarkably different from that obtained by the approach proposed in this paper. The differences are the ranking orders
between $A_{3}$ and $A_{5}$, and between $A_{1}$ and $A_{2}$, i.e., $A_{3} \succ A_{5}$ and $A_{1} \succ A$ for the former while $A_{5} \succ A_{3}$ and $A_{2} \succ A_{1}$ for the latter. Namely, the ranking orders of these pairs of alternatives are just converse. The main reason is that the our approach considers the hesitant fuzzy information which is represented by several possible values, not by a margin of error (as in HFEs), while if adopting Nan et al. (2008)'s method, it needs to transform HFEs into IFNs, which gives rise to a difference in the accuracy of data in the two types, it will have an effect on the final decision results. Thus it is not hard to see that Xu and Zhang (2013)'s approach has some desirable advantages over Nan et al. (2008)'s method as follows:
(1) Xu and Zhang (2013)'s approach, by extending the TOPSIS method to take into account the hesitant fuzzy assessments which are well-suited to handle the ambiguity and impreciseness inherent in the MADM problems, does not need to transform HFEs into IFNs but directly deals with these problems, and thus obtains better final decision results. In particular, when we meet some situations where the information is represented by several possible values, our approach shows its great superiority in handling those decision making problems with hesitant fuzzy information.
(2) Xu and Zhang (2013)'s approach utilizes the maximizing deviation method to objectively determine the weight values of attributes, which is more reasonable, while Nan et al. (2008)'s method needs the DM to provide the weight values in advance, which is subjective and sometime cannot yield the persuasive results.

### 4.2 ELECTRE I Method for Hesitant Fuzzy MADM

The ELECTRE method plays a significant role in the group of outranking. It was first set forth by Benayoun et al. (1966); Roy (1968), and referred as ELECTRE-I, which, along with its improvements, ELECTRE Iv and ELECTRE Is (Roy 1991; Roy and Skalka 1984), constitutes the so-called current version of the ELECTRE methods for choice problems. The approach is further developed as the ELECTRE-II, III and IV methods aiming at dealing with ranking problems, and as the ELECTRE-A and ELECTRE-TRI methods, devised to devote to sorting problems. See Figueira et al. (2005) for more details on these developments. Various forms of the ELECTRE methods have been widely applied to many fields (Figueira et al. 2005; Kaya and Kahraman 2011; Mousseau and Slowinski 1998), for example, project selection (Blondeau et al. 2002; Colson 2000), transportation (Roy et al. 1986), environment or water management (Lahdelma et al. 2002; Norese and Viale 2002), energy (Barda et al. 1990; Georgopoulou et al. 2003), agriculture and forest management (Duckstein and Gershon 1983; Srinivasa et al. 2000).

Furthermore, many authors (Fernandez and Olmedo 2005; Hokkanen and Salminen 1994, 1997; Leyva and Fernandez 2003; Rogers et al. 2000; Salminen et al. 1998) have applied the ELECTRE method to the field of group decision making (Bana e Costa 1986; Cabrerizo et al. 2009; Dias and Clímaco 1999, 2000, 2005; Kaya and Kahraman 2011; Kim and Ahn 1999; Kim and Han 1999; Lahdelma et al. 1998; Nazari-Shirkouhi et al. 2011). Dias and Clímaco (1999) computed

ELECTRE's credibility indices under partial information corresponding to the case when the DMs are unsure which values each parameter should take, which may result from insufficient, imprecise or contradictory information, as well as from different preferences among a group of DMs. Leyva and Fernandez (2003) presented an extension of the ELECTRE III multi-attribute outranking methodology to assist a group of DMs with different value systems to achieve a consensus on a set of possible alternatives. Fernandez and Olmedo (2005) proposed an agent model based on ideas of concordance and discordance for group ranking problems. Kaya and Kahraman (2011) illustrated an environmental impact assessment methodology based on an integrated fuzzy AHP-ELECTRE approach in the context of urban industrial planning. In addition, the ELECTRE method has recently been employed (El Hanandeh and El-Zein 2010; Hatami-Marbini and Tavana 2011; Montazer et al. 2009; Rogers and Bruen 1998; Sevkli 2009; Shanian et al. 2008; Siskos et al. 1984; Wu and Chen 2011; Vahdani et al. 2010) to handle the case that the evaluation information of the decision making problems may be uncertain and fuzzy, which is caused by the limited knowledge of DMs and the fuzzy nature of the real world. For instance, Hatami-Marbini and Tavana (2011) proposed the extended ELECTRE I method to account for the uncertain, imprecise and linguistic assessments provided by a group of DMs and used the mean value in aggregating the opinions of all DMs. Wu and Chen (2011) utilized the approach to solve the MCDM problems in intuitionistic fuzzy environments. Vahdani et al. (2010) applied it to assimilate the concepts of interval weights and data.

When performing group decision making by the frequently used approaches, the opinions of the DMs for each attribute and alternative are first aggregated and only a set of average attributes can be obtained, which implies a valid common decision. Thus these aggregation methods do not reflect differences between the individual DMs. Bana e Costa (1986), Kim and Ahn (1999), Kim and Han (1999) corrected the average sum aggregation of the performances in the framework of additive value functions. Dias and Clímaco (2005) designed a decision support system aiming at not imposing an aggregated model from the individual ones, but to reflect to each member the consequences of his/her inputs. Specifically, they dealt with the aggregation of multi-attribute performances by means of an additive value function under imprecise information using VIP analysis. The ELECTRE TRI methods (Dias and Clímaco 2000) were also employed for groups with imprecise information on parameter values.

The prominent features of the ELECTRE method include four binary relations, the preference modeling through outranking relations, and the concepts of concordance and discordance.

Definition 4.1 (Figueira et al. 2010). To compare two actions $a$ and $b$, binary relations are defined on the set $X$. For a pair $(a, b) \in X \times X$, let
(1) $K_{1}$ denotes the strict preference relation; and $a K_{1} b$ means that " $a$ is strictly preferred to $b^{\prime \prime}$;
(2) $K_{2}$ denotes the indifference relation, and $a K_{2} b$ means that " $a$ is indifferent to $b "$;
(3) $K_{3}$ denotes the weak preference relation, and $a K_{3} b$ means that " $a$ is weakly preferred to $b$, which expresses hesitation between the indifference ( $K_{2}$ ) and the preference ( $K_{1}$ ) ;
(4) $K_{4}$ denotes the incomparability relation, and $a K b$ means that " $a$ is not comparable to $b$ ". It corresponds to an absence of clear and positive reasons that would justify any of the three preceding relations.

Definition 4.2 (Figueira et al. 2010). Modeling preference in the ELECTRE method is via the comprehensive binary outranking relation $S$, whose meaning is "at least as good as"; In general, $S=K_{1} \cup K_{2} \cup K_{3}$. Considering two actions $(a, b) \in X \times X$, four cases appear:
(1) $a S b$ and not $b S a$, i.e., $a P b$ ( $a$ is strictly preferred to $b$ ).
(2) $b S a$ and not $a S b$, i.e., $b P a$ ( $b$ is strictly preferred to $a$ ).
(3) $a S b$ and $b S a$, i.e., $a I b$ ( $a$ is indifferent to $b$ ).
(4) Not $a S b$ and not $b S a$, i.e., $a K b$ ( $a$ is incomparable to $b$ ).

Definition 4.3 (Figueira et al. 2010). All outranking based methods rely on the concepts of concordance and discordance which represent, in a certain sense, the reasons for and against an outranking situation:
(1) The concordance concept: To validate an outranking $a S b$, a sufficient majority of criteria in favor of this assertion must occur.
(2) The discordance concept: The assertion $a S b$ cannot be validated if a minority of criteria is strongly against this assertion.

The ELECTRE method is composed of the construction and exploitation of one or several outranking relation(s) (Figueira et al. 2005). The construction is based on a comparison of each pair of actions on the attributers, through which concordance and discordance indices are obtained and they are further used to analyze the outranking relations among different alternatives.

In the traditional ELECTRE methods, each attribute in different alternatives can be divided into two different subsets: concordance set and discordance set. The former is composed of all attributes for which $A_{i}$ is preferred to $A_{j}$, and the latter is the complementary subset. However, in the hesitant fuzzy environments, according to the concepts of scores and deviation degrees, we can compare different alternatives on the attributes and classify different types of hesitant fuzzy concordance (discordance) sets as the hesitant fuzzy concordance (discordance) set and the weak hesitant fuzzy concordance (discordance) set. A better alternative has
the higher score or the lower deviation degree in cases where the alternatives have the same score.

As has been pointed out, HFEs can incorporate all possible opinions of the group members and thus provide intuitionistic descriptions on the differences among group members. In other words, when seeking a consensual solution in the process of group decision making, introducing HFEs can avoid aggregation (Torra and Narukawa 2009), that is, it does not need to force average preference on group members. Chen et al. (2013c) investigated the ELECTRE I method under hesitant fuzzy environments called hesitant fuzzy ELECTRE I (HF-ELECTRE I) method, which was primarily introduced to aid the process of group decision making. Moreover, in the method, they defined the concept of deviation function for HFEs to take the differences of opinions among group members into account.

In this section, we extend the ELECTRE I method to develop a new method in order to solve the MCDM problems under hesitant fuzzy environments.

As mentioned before, the set of all attributes is denoted as $X=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ and a HFS $A_{i}$ on $X$ is given by $A_{i}=\left\{<x_{j}, h_{A_{i}}\left(x_{j}\right)>\mid x_{j} \in X\right\}$, where $h_{A_{1}}\left(x_{j}\right)=\left\{\gamma \mid \gamma \in h_{A_{i}}\left(x_{j}\right), 0 \leq \gamma \leq 1\right\} \quad(i=1,2, \cdots, n ; j=1,2, \cdots, m)$ indicating all possible membership degrees of the $i$ th alternative $A_{i}$ under the $j$ th attribute $x_{j}$, expressed by a HFE $h_{i j}$.

For each pair of the alternatives $A_{i}$ and $A_{j}(i, j=1,2, \cdots, n, i \neq j)$, the hesitant fuzzy concordance set $J_{c_{i j}}$ of $A_{i}$ and $A_{j}$ is the sum of all those attributes where the performance of $A_{i}$ is superior to $A_{j}$. The hesitant fuzzy concordance set $J_{c_{i j}}$ can be formulated as follows:

$$
\begin{equation*}
J_{c_{i j}}=\left\{k \mid s\left(h_{i k}\right) \geq s\left(h_{j k}\right) \text { and } \bar{\sigma}\left(h_{i k}\right)<\bar{\sigma}\left(h_{j k}\right)\right\} \tag{4.19}
\end{equation*}
$$

The weak hesitant fuzzy concordance set $J_{c_{i j}}$ is defined by

$$
\begin{equation*}
J_{c_{i j}}=\left\{k \mid s\left(h_{i k}\right) \geq s\left(h_{j k}\right) \text { and } \bar{\sigma}\left(h_{i k}\right) \geq \bar{\sigma}\left(h_{j k}\right)\right\} \tag{4.20}
\end{equation*}
$$

The main difference between $J_{c_{i j}}$ and $J_{c_{i j}}$ lies in the deviation functions. The lower deviation values reflect that the opinions of DMs have a higher consistency degree. So $J_{c_{i j}}$ is more concordant than $J_{c_{i j}^{\prime}}$.

Similarly, the hesitant fuzzy discordance set $J_{d_{i j}}$, which is composed of all attributes for which $A_{i}$ is inferior to $A_{j}$, can be formulated as follows:

$$
\begin{equation*}
J_{d_{i j}}=\left\{k \mid s\left(h_{i k}\right)<s\left(h_{j k}\right) \text { and } \bar{\sigma}\left(h_{i k}\right) \geq \bar{\sigma}\left(h_{j k}\right)\right\} \tag{4.21}
\end{equation*}
$$

If $s\left(h_{i k}\right)<s\left(h_{j k}\right)$ and the deviation degree $\bar{\sigma}\left(h_{i k}\right)<\bar{\sigma}\left(h_{j k}\right)$, then we define this circumstance as the weak hesitant fuzzy discordance set $J_{d_{i j}^{\prime}}$, which is expressed by

$$
\begin{equation*}
J_{d_{i j}^{\prime}}=\left\{j \mid s\left(h_{i k}\right)<s\left(h_{j k}\right) \text { and } \bar{\sigma}\left(h_{i k}\right)<\bar{\sigma}\left(h_{j k}\right)\right\} \tag{4.22}
\end{equation*}
$$

Obviously, $J_{d_{i j}}$ is more discordant than $J_{d_{i j}^{\prime}}$.
In this subsection, we construct the corresponding matrices for different types of the hesitant fuzzy concordance sets and the hesitant fuzzy discordance sets, and propose the hesitant fuzzy ELECTRE I (HF-ELECTRE I) method. The hesitant fuzzy concordance index is the ratio of the sum of the weights related to attributes in the hesitant fuzzy concordance sets to that of all attributes. It can be computed by the values of the hesitant fuzzy concordance set in the HF-ELECTRE I method. The concordance index $c_{i j}$ of $A_{i}$ and $A_{j}$ in the HF-ELECTRE I method is defined as:

$$
\begin{equation*}
c_{i j}=\frac{\omega_{C} \times \sum_{k \in J_{c i j}} w_{k}+\omega_{C^{\prime}} \times \sum_{k \in J_{c i j}} w_{k}}{\sum_{k=1}^{n} w_{k}}=\omega_{C} \times \sum_{k \in J_{C i j}} w_{k}+\omega_{C^{\prime}} \times \sum_{k \in J_{c_{i j}}} w_{k} \tag{4.23}
\end{equation*}
$$

where $w_{k}$ is the weight of criterion and $\sum_{k=1}^{n} w_{k}=1$ for the normalized weight vector of all attreibutes. $\omega_{C}$ and $\omega_{C^{\prime}}$ are respectively the weights of the hesitant fuzzy concordance sets and the weak hesitant fuzzy concordance sets depending on the attitudes of the DMs. The $c_{i j}$ reflects the relative importance of $A_{i}$ with respect to $A_{j}$. Obviously, $0 \leq c_{i j} \leq 1$. A large value of $c_{i j}$ indicates that the alternative $A_{k}$ is superior to the alternative $A_{j}$. We can thus construct the asymmetrical hesitant fuzzy concordance matrix $C$ using the obtained value of the indices $c_{i j}(i, j=1,2, \cdots, n, \quad i \neq j)$, namely,

$$
C=\left(\begin{array}{cccccc}
- & \cdots & c_{1 j} & \cdots & c_{1(n-1)} & c_{1 n}  \tag{4.24}\\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
c_{i 1} & \cdots & c_{i j} & \cdots & c_{i(n-1)} & c_{i n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
c_{n 1} & \cdots & c_{n j} & \cdots & c_{n(n-1)} & -
\end{array}\right)
$$

Different from the hesitant fuzzy concordance index, the hesitant fuzzy discordance index is reflective of relative difference of $A_{i}$ with respect to $A_{j}$ in terms of discordance attributes. The discordance index is defined via:

$$
\begin{equation*}
d_{i j}=\frac{\max _{k \in J_{d_{i j}} \cup J_{d_{i j}}}\left\{\omega_{\bar{D}} \times d\left(w_{k} h_{i k}, w_{k} h_{j k}\right), \omega_{\bar{D}^{\prime}} \times d\left(w_{k} h_{i k}, w_{k} h_{j k}\right)\right\}}{\max _{k \in\{1,2, \ldots, n\}} d\left(w_{k} h_{i k}, w_{k} h_{j k}\right)} \tag{4.25}
\end{equation*}
$$

where $\omega_{\bar{D}}$ and $\omega_{\bar{D}^{\prime}}$ denote the weights of the hesitant fuzzy discordance set and the weak discordance set respectively depending on the DMs' attitudes, and $d\left(w_{k} h_{i k}, w_{k} h_{j k}\right)$ is the distance measure defined in Section 2.2.

Hesitant fuzzy discordance matrix is established by the hesitant fuzzy discordance index for all pairwise comparisons of alternatives:

$$
\bar{D}=\left(\begin{array}{cccccc}
- & \cdots & d_{1 j} & \cdots & d_{1(n-1)} & d_{1 n}  \tag{4.26}\\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
d_{i 1} & \cdots & d_{i j} & \cdots & d_{i(n-1)} & d_{i n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
d_{n 1} & \cdots & d_{n j} & \cdots & d_{n(n-1)} & -
\end{array}\right)
$$

As seen in Eqs.(4.23) and (4.24), the elements of $C$ differ substantially from those of $\bar{D}$, making the two matrices have complementary relationship, that is, the matrix $C$ represents the weights resulted from hesitant fuzzy concordance indices, whereas the asymmetrical matrix $\bar{D}$ reflects the relative difference of $w_{j} h_{i j}$ for all hesitant fuzzy discordance indices. Note that discordance matrix reflects the limited compensation between alternatives, that is, when the difference of two alternatives on an attribute arrives at a certain extent, compensation of the loss on a given attribute by a gain on another one may not be acceptable for the DMs (Figueira et al. 2005). Because of this reason discordance matrix is established differently from the case of the establishment of concordance matrix.

The hesitant fuzzy concordance dominance matrix can be calculated according to the cut-level of hesitant fuzzy concordance indices. If the hesitant fuzzy concordance index $c_{i j}$ of $A_{i}$ relative to $A_{j}$ is over a minimum level, then the superiority degree of $A_{i}$ to $A_{j}$ increases. The hesitant fuzzy concordance level can be defined as the average of all hesitant fuzzy concordance indices in the following manner:

$$
\begin{equation*}
\bar{c}=\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{c_{i j}}{n(n-1)} \tag{4.27}
\end{equation*}
$$

Based on the concordance level, the concordance dominance matrix $F$ (i.e., a Boolean matrix) can be expressed as:

$$
F=\left(\begin{array}{cccccc}
- & \cdots & f_{1 j} & \cdots & f_{1(n-1)} & f_{1 n}  \tag{4.28}\\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
f_{i 1} & \cdots & f_{i j} & \cdots & f_{i(n-1)} & f_{i n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
f_{n 1} & \cdots & f_{n j} & \cdots & f_{n(n-1)} & -
\end{array}\right)
$$

whose elements satisfy

$$
\begin{cases}f_{i j}=1, & \text { if } c_{i j} \geq \bar{c}  \tag{4.29}\\ f_{i j}=0, & \text { if } c_{i j}<\bar{c}\end{cases}
$$

where each element 1 in the matrix $F$ indicates that an alternative is preferable to the other one.

Likewise, the elements of the hesitant fuzzy discordance matrix are also measured by the discordance level $\bar{d}$, which can be defined as the average of the elements in the hesitant fuzzy discordance matrix:

$$
\begin{equation*}
\bar{d}=\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{d_{i j}}{n(n-1)} \tag{4.30}
\end{equation*}
$$

Then, based on the discordance level, the discordance dominance matrix $Q$ can be constructed as:

$$
Q=\left(\begin{array}{cccccc}
- & \cdots & q_{1 j} & \cdots & q_{1(n-1)} & q_{1 n}  \tag{4.31}\\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
q_{i 1} & \cdots & q_{i j} & \cdots & q_{i(n-1)} & q_{i n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
q_{n 1} & \cdots & q_{n j} & \cdots & q_{n(n-1)} & -
\end{array}\right)
$$

where

$$
\begin{cases}q_{i j}=1, & \text { if } d_{i j} \leq \bar{d}  \tag{4.32}\\ q_{i j}=0, & \text { if } d_{i j}>\bar{d}\end{cases}
$$

The elements of the matrix $Q$ measure the degree of the discordance. Hence, the discordant statement would be no longer valid if the element value $d_{i j} \leq \bar{d}$. That is to say, the elements of the matrix $Q$, whose values are 1 , show the dominant relations among the alternatives.

Then the aggregation dominance matrix $P$ is constructed from the elements of the matrix $F$ and the matrix $Q$ through the following formula:

$$
\begin{equation*}
P=F \otimes Q \tag{4.33}
\end{equation*}
$$

where each element $p_{i j}$ of $P$ is derived with

$$
\begin{equation*}
p_{i j}=f_{i j} q_{i j} \tag{4.34}
\end{equation*}
$$

Finally, we exploit the outranking relations aiming at elaborating recommendations from the results obtained in previous construction of the outranking relations (Roy and Skalka 1984).

If $p_{i j}=1$, then $A_{i}$ is strictly preferred to $A_{j}$ or $A_{i h}$ is weakly preferred to $A_{j}$; If $p_{i j}=1$ and $p_{j i}=1$, then $A_{i}$ is indifferent to $A_{j}$; If $p_{i j}=0$ and $p_{j i}=0$, then $A_{i}$ is incomparable to $A_{j}$.

We summarize the proposed HF-ELECTRE I method in the following steps and also display it in Fig. 4.3 (Chen et al. 2013c).

Step 1. Construct the hesitant fuzzy decision matrix. A group of DMs determine the relevant attributes of the potential alternatives and give the evaluation information in the form of HFEs of the alternative with respect to the attributes. They also determine the importance vector $w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}}$ for the relevant attributes, and the relative weight vector $\omega=\left(\omega_{C}, \omega_{C^{\prime}}, \omega_{\bar{D}}, \omega_{\bar{D}^{\prime}}\right)^{\mathrm{T}}$ of different types of the hesitant fuzzy concordance sets and the hesitant fuzzy discordance sets.

Step 2. Calculate the score function and the deviation function of each evaluation information of alternatives on the attributes in the form of HFEs according to Definition 1.2 and Eq.(1.2).

Step 3. Construct the hesitant fuzzy concordance sets and the hesitant fuzzy weak concordance sets using Eqs.(4.19) and (4.20).

Step 4. Construct the hesitant fuzzy discordance sets and the weak hesitant fuzzy discordance sets using Eqs.(4.21) and (4.22).

Step 5. Calculate the hesitant fuzzy concordance indexes using Eq.(4.23) and obtain the hesitant fuzzy concordance matrix using Eq.(4.24).

Step 6. Calculate the hesitant fuzzy discordance indexes using Eq.(4.24) and obtain the hesitant fuzzy discordance matrix using Eq.(4.25).

Step 7. Identify the concordance dominance matrix using Eqs.(4.26)-(4.27).

Step 8. Identify the discordance dominance matrix using Eqs.(4.28)-(4.30).
Step 9. Construct the aggregation dominance matrix using Eqs.(4.31)-(4.32).

Step 10. Draw a decision graph and choose the preferable alternative.
Step 11. End.
Now we use a numerical example to illustrate the details of the proposed HF-ELECTRE I method.

Example 4.2 (Chen et al. 2013c). The enterprise's board of directors, which includes five members, is to plan the development of large projects (strategy initiatives) for the following five years. Four possible projects $A_{i}(i=1,2,3,4)$ have been marked. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the balanced scorecard methodology (Kaplan and Norton 1996) (it should be noted that all of them are of the maximization type): (1) $x_{1}$ : Financial perspective; (2) $x_{2}$ : The customer satisfaction; (3) $x_{3}$ : Internal business process perspective; (4) $x_{4}$ : Learning and growth perspective.

In order to avoid psychic contagion, the DM are required to provide their preferences in anonymity. Suppose that the weight vector of the attributes is $w=(0.2,0.3,0.15,0.35)^{\mathrm{T}}$, and the hesitant fuzzy decision matrix is presented in the form of HFEs in Table 4.4 (Chen et al. 2013c).

| Step 1 <br> Construct the hesitant fuzzy decision matrix |  |
| :---: | :---: |
|  | 15 |
| Step 2Calculate the score function and deviation function |  |
|  | $\sqrt{1}$ |
| Step 3 <br> Construct the hesitant fuzzy concordance sets and the weak hesitant fuzzy concordance sets |  |
|  | 13 |
| Step 4 <br> Construct the hesitant fuzzy discordance sets and the weak discordance sets |  |
|  | 15 |
| Step 5 <br> Calculate the hesitant fuzzy concordance indexes and obtain the hesitant fuzzy concordance matrix |  |
|  | 15 |
| Step 6 <br> Calculate the hesitant fuzzy discordance indexes and obtain the hesitant fuzzy discordance matrix |  |
|  | 15 |
| Step 7 <br> Identify the concordance dominance matrix |  |
|  | 13 |
| Step 8 <br> Identify the discordance dominance matrix |  |
|  | 15 |
| Step 9 <br> Construct the aggregation dominance matrix |  |
|  | 15 |
| Step 10 <br> Draw a decision graph and choose the preferable alternative |  |

Fig. 4.3. The procedure of the HF-ELECTRE I method

Table 4.4. Hesitant fuzzy decision matrix for Example 4.2

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4,0.7\}$ | $\{0.2,0.6,0.8\}$ | $\{0.2,0.3,0.6,0.7,0.9\}$ | $\{0.3,0.4,0.5,0.7,0.8\}$ |
| $A_{2}$ | $\{0.2,0.4,0.7,0.9\}$ | $\{0.1,0.2,0.4,0.5\}$ | $\{0.3,0.4,0.6,0.9\}$ | $\{0.5,0.6,0.8,0.9\}$ |
| $A_{3}$ | $\{0.3,0.5,0.6,0.7\}$ | $\{0.2,0.4,0.5,0.6\}$ | $\{0.3,0.5,0.7,0.8\}$ | $\{0.2,0.5,0.6,0.7\}$ |
| $A_{4}$ | $\{0.3,0.5,0.6\}$ | $\{0.2,0.4\}$ | $\{0.5,0.6,0.7\}$ | $\{0.8,0.9\}$ |

Step 1. The hesitant fuzzy decision matrix and the weight vector of the attributes have been given above. The DMs also give the relative weights of the hesitant fuzzy concordance sets (weak hesitant fuzzy concordance sets) and the hesitant fuzzy discordance sets (weak discordance sets) respectively as follows:

$$
\omega=\left(\omega_{C}, \omega_{C^{\prime}}, \omega_{\bar{D}}, \omega_{\bar{D}^{\prime}}\right)^{\mathrm{T}}=\left(1, \frac{2}{3}, 1, \frac{2}{3}\right)^{\mathrm{T}}
$$

Step 2. Calculate the score and the deviation degree of each evaluation information of alternatives on the attributes in the form of HFEs according to Definition 1.2 and Eq.(1.2) represented in Tables 4.5-4.6 (Chen et al. 2013c):

Table 4.5. Scores

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.4333 | 0.5333 | 0.54 | 0.54 |
| $A_{2}$ | 0.55 | 0.3 | 0.55 | 0.7 |
| $A_{3}$ | 0.525 | 0.425 | 0.575 | 0.5 |
| $A_{4}$ | 0.4667 | 0.3 | 0.6 | 0.85 |

Table 4.6. Deviation degrees

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.2055 | 0.2494 | 0.2577 | 0.1855 |
| $A_{2}$ | 0.2693 | 0.1581 | 0.2291 | 0.1581 |
| $A_{3}$ | 0.1479 | 0.1479 | 0.1920 | 0.1871 |
| $A_{4}$ | 0.1247 | 0.1 | 0.0816 | 0.05 |

The data in Tables 4.5 and 4.6 are obtained from the data of Table 4.4. For example,

$$
\begin{gathered}
s\left(h_{21}\right)=\frac{0.2+0.4+0.7+0.9}{4}=0.55 \\
\sigma\left(h_{21}\right)=\left(\frac{1}{4}\left((0.2-0.55)^{2}+(0.4-0.55)^{2}+(0.7-0.55)^{2}+(0.9-0.55)^{2}\right)\right)^{\frac{1}{2}}=0.2693
\end{gathered}
$$

Step 3. Construct the hesitant fuzzy concordance sets and the weak hesitant fuzzy concordance sets:

$$
J_{C}=\left(\begin{array}{cccc}
- & - & \{4\} & - \\
\{3,4\} & - & \{4\} & - \\
\{1,3\} & \{2,3\} & - & - \\
\{1,3,4\} & \{2,3,4\} & \{3,4\} & -
\end{array}\right), \quad J_{C^{\prime}}=\left(\begin{array}{cccc}
- & \{2\} & \{2\} & \{2\} \\
\{1\} & - & \{1\} & \{1,2\} \\
- & - & - & \{1,2\} \\
- & - & - & -
\end{array}\right)
$$

For example, since

$$
s\left(h_{14}\right)=0.54>s\left(h_{34}\right)=0.5, \bar{\sigma}\left(h_{14}\right)=0.1855<\bar{\sigma}\left(h_{34}\right)=0.1871
$$

then $J_{c_{13}}=\{4\}$. Since

$$
s\left(h_{12}\right)=0.5333>s\left(h_{32}\right)=0.425
$$

and

$$
\bar{\sigma}\left(h_{12}\right)=0.2494>\bar{\sigma}\left(h_{32}\right)=0.1479
$$

then $J_{c_{13}^{\prime}}=\{2\}$.

Step 4. Construct the hesitant fuzzy discordance sets and the weak hesitant fuzzy discordance sets:

$$
J_{\bar{D}}=\left(\begin{array}{cccc}
- & 3,4 & 1,3 & 1,3,4 \\
- & - & 2,3 & 3,4 \\
4 & 4 & - & 3,4 \\
- & - & - & -
\end{array}\right), \quad J_{\bar{D}^{\prime}}=\left(\begin{array}{cccc}
- & 1 & - & - \\
2 & - & - & - \\
2 & 1 & - & - \\
2 & 1 & 1,2 & -
\end{array}\right)
$$

For example, since

$$
\begin{aligned}
& s\left(h_{13}\right)=0.54<s\left(h_{33}\right)=0.575, \bar{\sigma}\left(h_{13}\right)=0.2577>\bar{\sigma}\left(h_{33}\right)=0.1920 \\
& s\left(h_{11}\right)=0.4333<s\left(h_{31}\right)=0.525, \bar{\sigma}\left(h_{11}\right)=0.2055>\bar{\sigma}\left(h_{31}\right)=0.1479
\end{aligned}
$$

then $J_{d_{13}}=\{1,3\}$.

Step 5. Calculate the hesitant fuzzy concordance indexes and the hesitant fuzzy concordance matrix:

$$
C=\left(c_{i j}\right)_{4 \times 4}=\left(\begin{array}{cccc}
- & 0.2 & 0.55 & 0.2 \\
0.6333 & - & 0.4833 & 0.3333 \\
0.35 & 0.45 & - & 0.3333 \\
0.7 & 0.8 & 0.5 & -
\end{array}\right)
$$

For example, $c_{13}=\omega_{C} \times w_{4}+\omega_{C^{\prime}} \times w_{2}=1 \times 0.35+\frac{2}{3} \times 0.3=0.55$.
Step 6. Calculate the hesitant fuzzy discordance indexes and the hesitant fuzzy discordance matrix:

$$
\bar{D}=\left(d_{i j}\right)_{4 \times 4}=\left(\begin{array}{cccc}
- & 0.7778 & 0.3429 & 1 \\
0.6667 & - & 0.5357 & 1 \\
0.6667 & 1 & - & 1 \\
0.3361 & 0.3265 & 0.1143 & -
\end{array}\right)
$$

For instance,

$$
\begin{aligned}
d_{31} & =\frac{\max _{j \in J_{d_{31}} \cup J_{d_{31}}}\left\{\omega_{D} \times d\left(w_{j} h_{3 j}, w_{j} h_{1 j}\right), \omega_{D^{\prime}} \times d\left(w_{j} h_{3 j}, w_{j} h_{1 j}\right)\right\}}{\max _{j \in\{1,2,3,4\}} d\left(w_{j} h_{3 j}, w_{j} h_{1 j}\right)} \\
& =\frac{\max \left\{1 \times d\left(w_{4} h_{34}, w_{4} h_{14}\right), \frac{2}{3} \times d\left(w_{2} h_{32}, w_{2} h_{12}\right)\right\}}{0.0525} \\
& =\frac{\max \left\{1 \times 0.028, \frac{2}{3} \times 0.0525\right\}}{0.0525}=\frac{0.035}{0.0525}=0.6667
\end{aligned}
$$

where

$$
\begin{aligned}
& \begin{aligned}
d\left(w_{1} h_{31}, w_{1} h_{11}\right) & =\frac{1}{4} \times 0.2 \times(|0.2-0.3|+|0.4-0.5|+|0.7-0.6|+|0.7-0.7|) \\
& =0.015 \\
d\left(w_{2} h_{32}, w_{2} h_{12}\right) & =0.0525, \quad d\left(w_{3} h_{33}, w_{3} h_{13}\right)=0.018 \\
d\left(w_{4} h_{34}, w_{4} h_{14}\right) & =0.028
\end{aligned}
\end{aligned}
$$

Step 7. Calculate the hesitant fuzzy concordance level and identify the concordance dominance matrix, respectively:

$$
\bar{c}=\sum_{i=1}^{4} \sum_{j=1, j \neq i}^{4} \frac{c_{i j}}{4(4-1)}=0.4611, F=\left(\begin{array}{cccc}
- & 0 & 1 & 0 \\
1 & - & 1 & 0 \\
0 & 0 & - & 0 \\
1 & 1 & 1 & -
\end{array}\right)
$$

Step 8. Calculate the hesitant fuzzy discordance level and identify the discordance dominance matrix, respectively:

$$
\bar{d}=\sum_{i=1}^{4} \sum_{j=1, j \neq i}^{4} \frac{d_{i j}}{4(4-1)}=0.6472, Q=\left(\begin{array}{cccc}
- & 0 & 1 & 0 \\
0 & - & 1 & 0 \\
0 & 0 & - & 0 \\
1 & 1 & 1 & -
\end{array}\right)
$$

Step 9. Construct the aggregation dominance matrix:

$$
P=F \otimes Q=\left(\begin{array}{cccc}
- & 0 & 1 & 0 \\
0 & - & 1 & 0 \\
0 & 0 & - & 0 \\
1 & 1 & 1 & -
\end{array}\right)
$$

Step 10. As it can be seen from the aggregation dominance matrix, $A_{1}$ is preferred to $A_{3}, A_{2}$ is preferred to $A_{3}$ and $A_{4}$ is preferred to $A_{1}, A_{2}$ and $A_{3}$. Hence, $A_{4}$ is the best alternative. The results are depicted in Fig. 4.4 (Chen et al. 2013c).


Fig. 4.4. Decision graph of Example 4.2

The numerical example used here was also considered by Xia and Xu (2011a), who suggested the aggregation operators to fuse the hesitant fuzzy information and made the ranking of projects also by carrying out score functions for the same numerical example as ours. We find that both the average aggregation operators and the HF-ELECTRE I method give a consistent result, which clearly illustrates the validity of our proposed approach. These two proposed approaches are complementary when applied to solve different types of the MCDM problems. When the number of attributes in a MCDM problem is not larger than 4, the aggregation operator based approach is a suitable tool because of the simple solving steps. However, when the number of attributes exceeds 4 for hesitant fuzzy information, which often appears in some actual MCDM problems, the approach might encounter a barrier in applications because of the need for tremendous computation. For such cases, the proposed HF-ELECTRE I method is particularly useful, which is logically simple and demands less computational efforts. In what follows, we shall illustrate its application for the selection of investments where the number of criteria arrives at 6 .

Example 4.3. (Chen et al. 2013c) Assume that an enterprise wants to invest money in another country (adapted from Merigó and Casanovas (2011b)). A group composing of four DMs in the enterprise considers five possible investments: (1)
$A_{1}$ : Invest in the Asian market; (2) $A_{2}$ : Invest in the South American market; (3)
$A_{3}$ : Invest in the African market; (4) $A_{4}$ : Invest in the three continents; (5) $A_{5}$ : Do not invest in any continent.

Table 4.7. Hesitant fuzzy decision matrix for Example 4.3

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ | $A_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{1}$ | \{0.5,0.6,0.7\} | \{0,2,0.3, 0.6,0.7\} | \{0.3,0.4, 0.6 \} | \{0.2,0.3, 0,6\} | \{0.4,0.5, 0.6,0.7\} |
| $x_{2}$ | \{0.4,0.6,0.7,0.9\} | \{0.5,0.6,0.8\} | \{0.7,0.8\} | \{0.3,0.4, 0.6,0.7\} | \{0.7,0.9\} |
| $x_{3}$ | \{0.3,0.6,0.8,0.9\} | \{0.2,0.4,0.6 | \{0.3,0.6,0.7,0.9\} | \{0.2,0.4, 0.5, 0.6$\}$ | \{0.5,0.7, $0.8,0.9\}$ |
| $x_{4}$ | \{0.4,0.5, 0.6,0.8\} | \{0.3,0.6,0.7,0.8\} | \{0.2,0.3,0.5, 0.6\} | \{0.1,0.3, 0.4 \} | \{0.3,0.5, 0.6 |
| $x_{5}$ | \{0.2,0.5,0.6,0.7\} | \{0.3,0.6,0.8\} | \{0.3,0.4,0.7\} | \{0.2,0.3,0.7\} | \{0.4,0.5\} |
| $x_{6}$ | \{0.2,0.3,0.4, 0.6 \} | \{0.1,0.4\} | \{0.4,0.5,0.7\} | \{0.2,0.5,0.6\} | \{0.2,0.3,0.7\} |

When analyzing the investments, the DMs have considered the following general characteristics: (1) $x_{1}$ : Risks of the investment; (2) $x_{2}$ : Benefits in the short term; (3) $x_{3}$ : Benefits in the midterm; (4) $x_{4}$ : Benefits in the long term; (5) $x_{5}$ : Difficulty of the investment; (6) $x_{6}$ : Other aspects. After a careful analysis of these characteristics, the DMs have given the following information in the form of HFEs shown in Table 4.7 (Chen et al. 2013c).

Suppose that the weight vector of the attributes is $w=(0.25,0.2,0.15,0.1,0.2,0.1)^{\mathrm{T}}$, the DMs have also given the relative weights of the hesitant fuzzy concordance (weak hesitant fuzzy concordance) sets and the hesitant fuzzy discordance (weak discordance) sets as $\omega=\left(\omega_{C}, \omega_{C^{\prime}}, \omega_{D}, \omega_{D^{\prime}}\right)^{\mathrm{T}}=\left(1, \frac{3}{4}, 1, \frac{3}{4}\right)^{\mathrm{T}}$. Similar to the solving procedure used in the above numerical example, we obtain the aggregation dominance matrix:

$$
P=F \otimes Q=\left(\begin{array}{ccccc}
- & 1 & 0 & 1 & 0 \\
0 & - & 0 & 1 & 0 \\
0 & 0 & - & 1 & 0 \\
0 & 0 & 0 & - & 0 \\
0 & 1 & 0 & 1 & -
\end{array}\right)
$$

and a decision graph is constructed in Fig. 4.5 (Chen et al. 2013c).


Fig.4.5. Decision graph of Example 4.3

We can see from the graph that three preference relations are obtained. That is: (1) $A_{1} \succ A_{2} \succ A_{4}$; (2) $A_{5} \succ A_{2} \succ A_{4}$; (3) $A_{3} \succ A_{4}$. In contrast, if adopting the hesitant fuzzy average operators to aggregate the present hesitant information, the amount of data is extremely huge. For example, the number of HFE elements after aggregating $A_{1}$ reaches $3 \times 4^{5}=3072$. If aggregating all $A_{i}(i=1,2,3,4,5)$, then the corresponding number of computed data is 6677. The number will grow rapidly with increasing the number of alternatives and attributes. In our HF-ELECTR I method, the value for the number of calculation is 462. To determine the trends of the computation complexity for the two methods, we generate a great number of $n \times n$ hesitant fuzzy decision matrices $H=\left(h_{i j}\right)_{n \times n}$ (as an example, here we take the number of values for each $h_{i j}$ to be 4) by the Matlab optimization toolbox. The calculation times for the HF-ELECTRE I method and the aggregation operator methods are respective $\frac{1}{2}\left(5 n^{3}+9 n^{2}-10 n+4\right)$ and $n\left(4^{n}+1\right)$. To be more clear, we choose the cases of $n=4,5,10,15,20$ to demonstrate the trends of the computation complexity with increasing $n$ for the two methods. The results are given in Table 4.8 (Chen et al. 2013c).

Table 4.8. Calculation times for the HF-ELECTRE I method and the aggregation operator method

| Methods | $n=4$ | $n=5$ | $n=10$ | $n=15$ | $n=20$ |
| :---: | :---: | :---: | :---: | :---: | :--- |
| HF-ELECTRE I method | 214 | 402 | 2902 | 9377 | 21702 |
| Aggregation operator method | 1028 | 5125 | $1.05 \times 10^{7}$ | $1.61 \times 10^{10}$ | $2.19 \times 10^{13}$ |

In Example 4.3, only four alternatives are considered. To check possible influence arising from a change in the number of alternatives, we compare the outranking relation for different number of alternatives (i.e. $n=4,5,6$ and 7 ) under the same attributes. For this purpose, we perform a calculation with the Matlab Optimization Toolbox based on the HF-ELECTRE I method:
(1) $n=5$. We use the same calculation procedure as that in Example 4.3:

Step 1. Give the hesitant fuzzy decision matrix for the case of $n=5$ (Table 4.9 (Chen et al. 2013c)). Note that the former four alternatives are the same as those in Table 4.4 which corresponds to $n=4$.

Table 4.9. Hesitant fuzzy decision matrix for $n=5$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{0.2,0.4,0.7\}$ | $\{0.2,0.6,0.8\}$ | $\{0.2,0.3,0.6,0.7,0.9\}$ | $\{0.3,0.4,0.5,0.7,0.8\}$ |
| $A_{2}$ | $\{0.2,0.4,0.7,0.9\}$ | $\{0.1,0.2,0.4,0.5\}$ | $\{0.3,0.4,0.6,0.9\}$ | $\{0.5,0.6,0.8,0.9\}$ |
| $A_{3}$ | $\{0.3,0.5,0.6,0.7\}$ | $\{0.2,0.4,0.5,0.6\}$ | $\{0.3,0.5,0.7,0.8\}$ | $\{0.2,0.5,0.6,0.7\}$ |
| $A_{4}$ | $\{0.3,0.5,0.6\}$ | $\{0.2,0.4\}$ | $\{0.5,0.6,0.7\}$ | $\{0.8,0.9\}$ |
| $A_{5}$ | $\{0.4,0.6,0.7,0.9\}$ | $\{0.3,0.4,0.6\}$ | $\{0.3,0.4,0.5,0.8\}$ | $\{0.6,0.7,0.9\}$ |

Step 2. Calculate the scores and the deviation degrees of evaluation information of the alternative $A_{5}$ on the criteria in the form of HFEs listed in Table 4.10 (Chen et al. 2013c).

Table 4.10. Score values and deviation values of $A_{5}$

| $A_{5}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| Scores | 0.65 | 0.4333 | 0.5 | 0.7333 |
| deviation degrees | 0.1803 | 0.1247 | 0.1871 | 0.1247 |

Note that for the former four alternatives (i.e. $A_{1}, A_{2}, A_{3}, A_{4}$ ), their scores and deviations of evaluation information are the same as those in Example 4.3, and thus, they are not repeated in Table 4.10.

Step 3. Construct the hesitant fuzzy concordance sets and the weak hesitant fuzzy concordance sets:

$$
\begin{gathered}
J_{C}=\left(\begin{array}{ccccc}
- & - & \{4\} & - & - \\
\{3,4\} & - & \{4\} & - & - \\
\{1,3\} & \{2,3\} & - & - & - \\
\{1,3,4\} & \{2,3,4\} & \{3,4\} & - & \{3,4\} \\
\{1,4\} & \{1,2,4\} & \{2,4\} & - & -
\end{array}\right) \\
J_{C^{\prime}}=\left(\begin{array}{ccccc}
- & \{2\} & \{2\} & \{2\} & \{2,3\} \\
\{1\} & - & \{1\} & \{1,2\} & \{3\} \\
- & - & - & \{1,2\} & \{3\} \\
- & - & - & - & - \\
- & - & 1 & \{1,2\} & -
\end{array}\right)
\end{gathered}
$$

Step 4. Construct the hesitant fuzzy discordance sets and the weak hesitant fuzzy discordance sets:

$$
\begin{aligned}
J_{\bar{D}}= & \left(\begin{array}{ccccc}
- & \{3,4\} & \{1,3\} & \{1,3,4\} & \{1,4\} \\
- & - & \{2,3\} & \{3,4\} & \{1,2,4\} \\
\{4\} & \{4\} & - & \{3,4\} & \{2,4\} \\
- & - & - & - & - \\
- & - & - & \{3,4\} & -
\end{array}\right) \\
J_{\bar{D}^{\prime}} & =\left(\begin{array}{ccccc}
- & \{1\} & - & - & - \\
\{2\} & - & - & - & - \\
\{2\} & \{1\} & - & - & \{1\} \\
\{2\} & \{1\} & \{1,2\} & - & \{1,2\} \\
\{2,3\} & \{3\} & \{3\} & - & -
\end{array}\right)
\end{aligned}
$$

Step 5. Calculate the hesitant fuzzy concordance indices and the hesitant fuzzy concordance matrix:

$$
C=\left(c_{i j}\right)_{5 \times 5}=\left(\begin{array}{ccccc}
- & 0.2 & 0.55 & 0.2 & 0.3 \\
0.6333 & - & 0.4833 & 0.3333 & 0.1 \\
0.35 & 0.45 & - & 0.3333 & 0.1 \\
0.7 & 0.8 & 0.5 & - & 0.5 \\
0.55 & 0.85 & 0.7833 & 0.3333 & -
\end{array}\right)
$$

Step 6. Calculate the hesitant fuzzy discordance indices and the hesitant fuzzy discordance matrix:

$$
D=\left(d_{i j}\right)_{5 \times 5}=\left(\begin{array}{ccccc}
- & 0.7778 & 0.3429 & 1 & 1 \\
0.6667 & - & 0.5357 & 1 & 1 \\
0.6667 & 1 & - & 1 & 1 \\
0.3361 & 0.3265 & 0.1143 & - & 0.4285 \\
0.3663 & 0.0952 & 0.0779 & 1 & -
\end{array}\right)
$$

As seen from Step 3 to Step 6, adding the alternative $A_{5}$ does not change the hesitant fuzzy concordance (discordance) sets, the weak hesitant fuzzy concordance (discordance) sets, the hesitant fuzzy concordance (discordance) indices, the hesitant fuzzy concordance (discordance) matrix of the former four alternatives. Its role is to add the last column and the last line of the resulting corresponding matrices. Similar situations also appear when the number of the alternatives is $n=6$ and $n=7$.

Step 7. Calculate the hesitant fuzzy concordance level and identify the concordance dominance matrix, respectively:

$$
\bar{c}=\sum_{i=1}^{5} \sum_{j=1, j \neq i}^{5} \frac{c_{i j}}{n(n-1)}=0.4525, \quad F=\left[\begin{array}{ccccc}
- & 0 & 1 & 0 & 0 \\
1 & - & 1 & 0 & 0 \\
0 & 0 & - & 0 & 0 \\
1 & 1 & 1 & - & 1 \\
1 & 1 & 1 & 0 & -
\end{array}\right]
$$

Step 8. Calculate the hesitant fuzzy discordance level and identify the discordance dominance matrix, respectively:

$$
\bar{d}=\sum_{i=1}^{5} \sum_{j=1, j \neq i}^{5} \frac{d_{i j}}{5(5-1)}=0.6367, Q=\left(\begin{array}{ccccc}
- & 0 & 1 & 0 & 0 \\
0 & - & 1 & 0 & 0 \\
0 & 0 & - & 0 & 0 \\
1 & 1 & 1 & - & 1 \\
1 & 1 & 1 & 0 & -
\end{array}\right)
$$

Step 9. Construct the aggregation dominance matrix:

$$
P=F \otimes Q=\left(\begin{array}{ccccc}
- & 0 & 1 & 0 & 0 \\
0 & - & 1 & 0 & 0 \\
0 & 0 & - & 0 & 0 \\
1 & 1 & 1 & - & 1 \\
1 & 1 & 1 & 0 & -
\end{array}\right)
$$

Step 10. As it can be seen from the aggregation dominance matrix that $A_{1}$ is preferred to $A_{3} ; A_{2}$ is preferred to $A_{3} ; A_{4}$ is preferred to $A_{1} A_{2}, A_{3}$ and $A_{5} ; A_{5}$ is preferred to $A_{1}, A_{2}$ and $A_{3}$. The results are depicted in Fig. 4.6 (Chen et al. 2013c).


Fig. 4.6. Decision graph of five alternatives
(2) $n=6$. Results involving only the alternative $A_{6}$ obtained in Steps 1 and 2 are listed in Tables 4.11 and 4.12 (Chen et al. 2013c), respectively.

Table 4.11. Hesitant fuzzy decision matrix for $A_{6}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{6}$ | $\{0.5,0.7,0.9\}$ | $\{0.4,0.5,0.7,0.9\}$ | $\{0.5,0.8\}$ | $\{0.4,0.5,0.8\}$ |

Table 4.12. Score values and deviation values of $A_{6}$

| $A_{6}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| Score values | 0.7 | 0.625 | 0.65 | 0.5667 |
| deviation values | 0.1633 | 0.1920 | 0.15 | 0.1700 |

Table 4.13. Data of $J_{c_{i j}}, J_{c_{i j}^{\prime}}, J_{d_{i j}}, J_{d_{i j}^{\prime}}, c_{i j}$ and $d_{i j}$ of the 6th line and the 6th column for $n=6$

|  | $J_{c_{i j}}$ | $J_{c_{i j}}$ | $J_{d_{i j}}$ | $J_{d_{i j}}$ | $c_{i j}$ | $d_{i j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i=1, j=6$ | - | - | $\{1,2,3,4\}$ | - | 0 | 1 |
| $i=2, j=6$ | $\{4\}$ | - | $\{1,3\}$ | $\{2\}$ | 0.35 | 0.6667 |
| $i=3, j=6$ | - | - | $\{3,4\}$ | $\{1,2\}$ | 0 | 0.7292 |
| $i=4, j=6$ | $\{4\}$ | - | - | $\{1,2,3\}$ | 0.35 | 0.5238 |
| $i=5, j=6$ | $\{4\}$ | - | $\{1,3\}$ | $\{2\}$ | 0.35 | 0.5786 |
| $i=6, j=1$ | $\{1,2,3,4\}$ | - | - | - | 1 | 0 |
| $i=6, j=2$ | $\{1,3\}$ | $\{2\}$ | $\{4\}$ | - | 0.55 | 0.2692 |
| $i=6, j=3$ | $\{3,4\}$ | $\{1,2\}$ | - | - | 0.8333 | 0 |
| $i=6, j=4$ | - | $\{1,2,3\}$ | $\{4\}$ | - | 0.4333 | 1 |
| $i=6, j=5$ | $\{1,3\}$ | $\{2\}$ | $\{4\}$ | - | 0.55 | 1 |

When the number of alternatives arrives at 6, the results obtained from Step 3 to Step 6 can be expressed with the $6 \times 6$ matrix in which except the data of the 6 th line and the 6th column, other data are the same as those in $5 \times 5$ matrix when the number of alternatives is 5 . Thus, we only list in Table 4.13 (Chen et al. 2013c) the data of the 6 th line and the 6 th column of the $6 \times 6$ matrix, which are the hesitant fuzzy concordance set $J_{c_{i j}}$, the weak hesitant fuzzy concordance set $J_{c_{i j}^{\prime}}$, the weak hesitant fuzzy discordance set $J_{d_{i j h}}$, the hesitant fuzzy discordance set $J_{d_{i j}^{\prime}}$, the hesitant fuzzy concordance indexes $c_{i j}$, and the hesitant fuzzy discordance indexes $d_{i j}$, respectively.

Steps 7 and 8. Calculate the hesitant fuzzy concordance level and the discordance level:

$$
\bar{c}=\sum_{k=1}^{m} \sum_{l=1, l \neq k}^{m} \frac{c_{k l}}{m(m-1)}=0.4489, \bar{d}=\sum_{k=1}^{m} \sum_{l=1, l \neq k}^{m} \frac{d_{k l}}{m(m-1)}=0.6167
$$

Step 9. Construct the aggregation dominance matrix:

$$
P=\left(\begin{array}{cccccc}
- & 0 & 1 & 0 & 0 & 0 \\
0 & - & 1 & 0 & 0 & 0 \\
0 & 0 & - & 0 & 0 & 0 \\
1 & 1 & 1 & - & 1 & 0 \\
1 & 1 & 1 & 0 & - & 0 \\
1 & 1 & 1 & 0 & 0 & -
\end{array}\right)
$$

Step 10. As it can be seen from the matrix $P$ that $A_{1}$ is preferred to $A_{3} ; A_{2}$ is preferred to $A_{3} ; A_{4}$ is preferred to $A_{1}, A_{2}, A_{3}$ and $A_{5} ; A_{5}$ is preferred to $A_{1}$, $A_{2}$ and $A_{3} ; A_{6}$ is preferred to $A_{1}, A_{2}$ and $A_{3}$ (see Fig. 4.7 (Chen et al. 2013c)).


Fig. 4.7. Decision graph of six alternatives
(3) $n=7$. Analogous to the case of $n=6$, main results obtained from Step 1 to Step 6 are summarized in Tables 4.13-4.15 (Chen et al. 2013c).

Table 4.14. Hesitant fuzzy decision matrix for $A_{7}$

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{7}$ | $\{0.2,0.3,0.4\}$ | $\{0.2,0.3,0.4,0.7\}$ | $\{0.3,0.5,0.6,0.7\}$ | $\{0.2,0.3,0.4,0.5\}$ |

Table 4.15. Scores and deviations of $A_{7}$

| $A_{7}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| Scores | 0.3 | 0.4 | 0.525 | 0.35 |
| deviation degrees | 0.0816 | 0.1871 | 0.1479 | 0.1118 |

Table 4.16. Data of $J_{c_{i j}}, J_{c_{i j}^{\prime}}, J_{d_{i j}}, J_{d_{i j}^{\prime}}, c_{i j}$ and $d_{i j}$ of the 7 th line and the 7 th column for $n=7$

|  | $J_{c_{i j}}$ | $J_{c_{i j}}$ | $J_{d_{i j}}$ | $J_{d_{i j}^{\prime}}$ | $c_{i j}$ | $d_{i j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i=1, j=7$ | - | $\{1,2,3,4\}$ | - | - | 0.6667 | 0 |
| $i=2, j=7$ | - | $\{1,3,4\}$ | - | $\{2\}$ | 0.4667 | 0.1633 |
| $i=3, j=7$ | $\{2\}$ | $\{1,3,4\}$ | - | - | 0.7667 | 0 |
| $i=4, j=7$ | $\{3,4\}$ | $\{1\}$ | - | $\{2\}$ | 0.6333 | 0.1088 |
| $i=5, j=7$ | $\{2\}$ | $\{1,4\}$ | $\{3\}$ | - | 0.6667 | 0.0756 |
| $i=6, j=7$ | - | $\{1,2,3,4\}$ | - | - | 0.6667 | 0 |
| $i=7, j=1$ | - | - | - | $\{1,2,3,4\}$ | 0 | 0.6667 |
| $i=7, j=2$ | - | $\{2\}$ | - | $\{1,3,4\}$ | 0.2 | 0.6667 |
| $i=7, j=3$ | - | - | $\{2\}$ | $\{1,3,4\}$ | 0 | 0.6667 |
| $i=7, j=4$ | - | $\{2\}$ | $\{3,4\}$ | $\{1\}$ | 0.2 | 1 |
| $i=7, j=5$ | $\{3\}$ | - | $\{2\}$ | $\{1,4\}$ | 0.15 | 0.6667 |
| $i=7, j=6$ | - | - | - | $\{1,2,3,4\}$ | 0 | 0.6667 |

Steps 7 and 8. Calculate the hesitant fuzzy concordance level and discordance level:

$$
\bar{c}=\sum_{i=1}^{7} \sum_{j=1, j \neq i}^{7} \frac{c_{i j}}{7(7-1)}=0.4258, \bar{d}=\sum_{i=1}^{7} \sum_{j=1, j \neq i}^{7} \frac{d_{i j}}{7(7-1)}=0.5520
$$

Step 9. Construct the aggregation dominance matrix:

$$
P=\left(\begin{array}{ccccccc}
- & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & - & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & - & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & - & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & - & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -
\end{array}\right)
$$

Step 10. As it can be seen from the aggregation dominance matrix that $A_{1}$ is preferred to $A_{3}$ and $A_{7} ; A_{2}$ is preferred to $A_{3}$ and $A_{7} ; A_{3}$ is preferred to $A_{7} ; A_{4}$ is preferred to $A_{1}, A_{2}, A_{3}, A_{5}$ and $A_{7} ; A_{5}$ is preferred to $A_{1}$, $A_{2}, A_{3}$ and $A_{7} ; A_{6}$ is preferred to $A_{1}, A_{2} A_{3}$ and $A_{7}$ (see Fig. 4.8 (Chen et al. 2013c)).


Fig. 4.8. Decision graph of seven alternatives

To facilitate to see the possible influence due to a change in the number of alternatives, the outranking relations obtained for the cases of $N=4,5,6$ and 7 are listed in Table 4.16 (Chen et al. 2013c).

Table 4.17. Comparison of the outranking relations for different number of alternatives

| The number of alternatives | $\bar{c}$ | $\bar{d}$ | Results |
| :---: | :---: | :---: | :---: |
| $n=4$ <br> (Example 4.3) | 0.4611 | 0.6472 | $\begin{gathered} A_{1} \succ A_{3} ; A_{2} \succ A_{3} ; \\ A_{4} \succ A_{1}, A_{2}, A_{3} \end{gathered}$ |
| $n=5$ | 0.4525 | 0.6367 | $\begin{gathered} A_{1} \succ A_{3} ; A_{2} \succ A_{3} ; \\ A_{4} \succ A_{1}, A_{2}, A_{3}, A_{5} ; \\ A_{5} \succ A_{1}, A_{2}, A_{3} \end{gathered}$ |
| $n=6$ | 0.4489 | 0.6167 | $\begin{gathered} A_{1} \succ A_{3} ; A_{2} \succ A_{3} ; \\ A_{4} \succ A_{1}, A_{2}, A_{3}, A_{5} ; \\ A_{5} \succ A_{1}, A_{2}, A_{3} ; \\ A_{6} \succ A_{1}, A_{2}, A_{3} \end{gathered}$ |
| $n=7$ | 0.4258 | 0.5520 | $\begin{aligned} & A_{1} \succ A_{3}, A_{7} ; A_{2} \succ A_{3}, A_{7} ; \\ & A_{3} \succ A_{7} ; A_{4} \succ A_{1}, A_{2}, A_{3}, A_{5}, A_{7} ; \\ & A_{5} \succ A_{1}, A_{2}, A_{3}, A_{7} ; \\ & A_{6} \succ A_{1}, A_{2}, A_{3}, A_{7} \end{aligned}$ |

It can be seen from Table 4.17 (Chen et al. 2013c) that varying the number of alternatives does not change the result that $A_{4}$ is a non-outranked alternative in Example 4.3. We explain it in the following way:

First, when the number of alternatives is respectively 5, 6 and 7, the hesitant fuzzy concordance (discordance) indices given in Example 4.3 do not be changed. Second, although a variation in the number of alternatives slightly modifies the hesitant fuzzy concordance level $\bar{c}$ and the discordance level $\bar{d}$ (see Table 4.17 (Chen et al. 2013c)), but the changes in $\bar{c}$ and $\bar{d}$ are still within the sensitivity range of Example 4.3 in which the parameter changes will not affect the set of the non-outranked alternatives. Specifically, in Example 4.3, $\bar{c}=0.4611$ and $\bar{d}=0.6472$. A decrease (increase) in $\bar{c}(\bar{d})$ cannot bring about a change in the set of alternatives that are not outranked by other alternatives (Vetschera 1986). Thus, $\bar{c}$ could be lowered to zero and $\bar{d}$ could be raised to 1 . Moreover, we note
from the hesitant fuzzy concordance (discordance) matrices (given in Step 5 (Step 6) in Example 4.3) that when $\bar{c}$ is increased above 0.55, the alternative $A_{3}$ is no longer outranked by any other alternative. For the alternative $A_{1}\left(A_{2}\right)$, the corresponding value is $0.7(0.8)$, as for $\bar{c}<0.7(\bar{c}<0.8)$ it remains outranked by the alternative $A_{4}$. The first change in the set of the non-outranked alternatives occurs when $\bar{c}$ is increased to 0.55 . A similar analysis can be performed for the discordance level. When $\bar{d}$ is lowered below $0.3361, A_{1}$ is no longer outranked. The corresponding value for $A_{2}\left(A_{3}\right)$ is $0.3265(0.1143)$, so the lower bound for $\bar{d}$ is thus given by 0.3361 . Taken together, we find in Example 4.3 that when the parameters satisfy $0<\bar{c}<0.55$ and $0.3361<\bar{d}<1$, the set of the non-outranked alternatives will not be affected.

In the following, we survey the outranking relations for different numbers of attributes. To this end we increase the number of attributes by five and six (see Table 4.18 (Chen et al. 2013c)).

Table 4.18. Hesitant fuzzy information of the fifth and sixth attribute for each alternative

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{5}$ |
| :---: | :---: | :---: | :---: | :---: |
| $x_{5}$ | $\{0.3,0.6,0.7\}$ | $\{0.2,0.4,0.5,0.6\}$ | $\{0.3,0.7,0.8\}$ | $\{0.4,0.5,0.7,0.9\}$ |
| $x_{6}$ | $\{0.4,0.5,0.7,0.8\}$ | $\{0.5,0.7,0.9\}$ | $\{0.3,0.4,0.6,0.9\}$ | $\{0.5,0.6,0.8\}$ |

Because the weights of all attributes satisfy the normalization constraint $\sum_{j=1}^{n} w_{j}=1$, varying the number of attributes inevitably changes the original weights of attributes. This will affect the outranking relations, as have been pointed out in many previous works (see, e.g., Vetschera (1986), Henggeler Antunes and Clímaco 1992). As an illustration, we list in Table 4.19 (Chen et al. 2013c) the corresponding results for the outranking relations within the HF-ELECTRE I framework and the comparison with the case that the number of attributes is four. As it can be seen in the table, the results of the outranking relations for different number of attributes calculated with the HF-ELECTRE I method are consistent with the general expectation.

Table 4.19. Comparison of the outranking relations for different number of attributes

| The number of <br> attributes | Weight | Results |
| :---: | :---: | :---: |
| 4 |  |  |
| (Example 4.3) | $w=(0.2,0.3,0.15,0.35)^{\mathrm{T}}$ | $A_{1} \succ A_{3} ; A_{2} \succ A_{3} ;$ <br> $A_{4} \succ A_{1}, A_{2}, A_{3}$ <br> 5 |
|  | $w=(0.2,0.3,0.15,0.2,0.15)^{\mathrm{T}}$ | $A_{2} \succ A_{1} ; A_{3} \succ A_{1} ;$ |
|  | $A_{4} \succ A_{1}, A_{2}, A_{3}$ |  |
| 6 | $w=(0.2,0.15,0.15,0.2,0.15,0.15)$ | $A_{4} \succ A_{1}, A_{2}, A_{3} ;$ <br> $A_{2} \succ A_{1}, A_{3} ; A_{3} \succ A_{1}$ |

In order to further evaluate the HF-ELECTRE I method, a simulation with randomly generated cases is made in a direct and transparent way. Random data are generated to form the MCDM problems with all possible combinations of four, six, eigth, ten alternatives and four, six, eight, ten attributes. So, sixteen $(4 \times 4)$ different instances are examined in this study. We find that the alternative $A_{1}$ (see below for details) in all these instances is the best one.

In the following, the cases corresponding to the combinations of four alternatives (i.e., $A=\left\{A_{1}, A_{2}, A_{3}, A_{4}\right\}$ ) with four, six, eight, ten attributes are employed to illustrate our simulation process. Under the environment of group decision making, the performance of each alternative on each attribute can be considered as a HFS, represented by $A_{i}=\left\{\left\langle x_{j}, h_{A_{i}}\left(x_{j}\right)\right\rangle \mid x_{j} \in X\right\}, \quad i=1, \cdots, 4$, where $h_{Y_{i}}\left(x_{j}\right)\left(=\left\{\gamma \mid \gamma \in h_{A_{i}}\left(x_{j}\right), 0 \leq \gamma \leq 1\right\}\right)$ indicates all possible membership degrees of the alternative $A_{i}$ on the attribute $x_{j}$. The characteristics of these four alternatives are assigned as follows: For $\gamma \in h_{A_{1}}\left(x_{j}\right), \quad \gamma \sim \dot{U}(0.8,1)$; For $\gamma \in h_{A_{2}}\left(x_{j}\right), \gamma \sim \dot{U}(0.6,0.8)$; For $\gamma \in h_{A_{3}}\left(x_{j}\right), \quad \gamma \sim \dot{U}(0.4,0.6)$; For $\gamma \in h_{A_{4}}\left(x_{j}\right), \gamma \sim \dot{U}(0.1,0.4)$, where $\dot{U}(a, b)$ means the uniform distribution on the interval $[a, b]$.

In terms of the above-described way, we have simulated 100 times corresponding to each instance with the number of criteria being four, six, eight and ten, respectively. Besides the $\gamma$ values, in each simulation, the number of $\gamma$ in
each $h_{A_{i}}\left(x_{j}\right)$ and the weights of all attributes (which satisfy $0 \leq w_{j} \leq 1$, $j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$ ) are also randomly generated. Consequently, all quantities calculated in the HF-ELECTRTE I fluctuate in each time simulation. As an example, Fig. 4.9 (Chen et al. 2013c) displays the scores of $h_{A_{i}}\left(x_{j}\right)$ of four alternatives on an attribute, which shows the obvious variations for different simulations.


Fig. 4.9. Scores of $h_{A_{i}}\left(x_{j}\right)$ of the alternative $A_{1}$ (yellow line), $A_{2}$ (blue line), $A_{3}$ (green line) and $A_{4}$ (red line) on an attribute in a hundred times simulations

Following the steps of the HF-ELECTRE I method outlined previously, our simulation results for the randomly generated instances demonstrate that the outranking relations are consistent with the expectation, that is, $A_{1}$ is the best alternative due to a larger role of the performance of alternatives in influencing the outranking relations as compared to the weights and threshold values (Buchanan and Vanderpooten 2007). This consistency further indicates the validity of the HF-ELECTRE I method.

As we know, there exist several other outranking ELECTRE type methods, for example, ELECTRE III (Buchanan and Vanderpooten 2007; Figueira et al. 2005) and ELECTRE IV (Roy and Hugonnard 1982) methods, which are able to deal with the ranking problems, that is, it is concerned with the ranking of all the actions belonging to a given set of actions from the best to the worst (Figueira et al. 2005). Although the HF-ELECTRE I method is used to construct a partial prioritization
and to choose a set of preferable actions, which is somewhat different from the ELECTRE III and ELECTRE IV methods, it is interesting to compare the results obtained using these different methods aiming at seeing the ranking differences among them:

To facilitate a comparison, the same example (Example 4.3) used to illustrate the HF-ELECTRE I method is also considered for the ELECTRE III and ELECTRE IV methods, which is composed of the construction and the exploitation of the outranking relations.

We first briefly discuss the ELECTRE III method, for more details see Buchanan and Vanderpooten (2007), and Figueira et al. (2005).

We introduce the construction of the outranking relations. Let two alternatives $A_{1}$ and $A_{2}$ belong to a given set of actions, and $y_{j}\left(A_{1}\right)$ and $y_{j}\left(A_{2}\right)$ the performances of $A_{1}$ and $A_{2}$ in terms of the attribute $x_{j}$. We denote indifference, preference and veto thresholds on an attribute $x_{j}$ introduced by the DMs with $q_{j}, \quad p_{j}$ and $o_{j}$, respectively, where $j=1,2, \cdots, n$.

A partial concordance index $c_{j}\left(A_{1}, A_{2}\right)$ is defined as follows for each attribute $x_{j}$ :

$$
c_{j}\left(A_{1}, A_{2}\right)=\left\{\begin{array}{cc}
1, & \text { if } y_{j}\left(A_{1}\right)+q_{j} \geq y_{j}\left(A_{2}\right)  \tag{4.35}\\
0, & \text { if } y_{j}\left(A_{1}\right)+p_{j} \leq y_{j}\left(A_{2}\right), \\
\frac{p_{j}+y_{j}\left(A_{1}\right)-y_{j}\left(A_{2}\right)}{p_{j}-q_{j}}, & \text { otherwise }
\end{array} \quad j=1, \cdots, n\right.
$$

Let $w_{j}$ be the importance coefficient for the attribute $x_{j} . c\left(A_{1}, A_{2}\right)$ is an overall concordance index and defined as:

$$
\begin{equation*}
c\left(A_{1}, A_{2}\right)=\frac{1}{\bar{w}} \sum_{j=1}^{n} w_{j} c_{j}\left(A_{1}, A_{2}\right) \tag{4.36}
\end{equation*}
$$

where $\bar{w}=\sum_{j=1}^{n} w_{j}$. The discordance index for each attribute $x_{j}, d_{j}\left(A_{1}, A_{2}\right)$ is calculated as:

$$
d_{j}\left(A_{1}, A_{2}\right)=\left\{\begin{array}{cc} 
& \text { if } y_{j}\left(A_{1}\right)+p_{j} \geq y_{j}\left(A_{2}\right)  \tag{4.37}\\
0, & \text { if } y_{j}\left(A_{1}\right)+o_{j} \leq y_{j}\left(A_{2}\right), \\
1, & \text { otherwise } \\
\frac{y_{j}\left(A_{2}\right)-y_{j}\left(A_{1}\right)-p_{j}}{o_{j}-p_{j}}, & j=1, \cdots, n\}
\end{array}\right.
$$

The credibility degree $\bar{S}\left(A_{1}, A_{2}\right)$ for each pair $\left(A_{1}, A_{2}\right)$ is defined as:

$$
\bar{S}\left(A_{1}, A_{2}\right)=\left\{\begin{array}{ccc}
c\left(A_{1}, A_{2}\right), & \text { if } d_{j}\left(A_{1}, A_{2}\right) \leq c\left(A_{1}, A_{2}\right), \forall j  \tag{4.38}\\
c\left(A_{1}, A_{2}\right) \cdot & \prod_{j \in[1,2, \ldots, n\} d_{j}\left(A_{1}, A_{2}\right)>c\left(A_{1}, A_{2}\right)} \frac{1-d_{j}\left(A_{1}, A_{2}\right)}{c\left(A_{1}, A_{2}\right)}, & \text { otherwise }
\end{array}\right.
$$

The ELECTRE model is "exploited" to produce a project ranking from the credibility matrix. We follow the standard implementation presented by Buchanan and Vanderpooten (2007), and Figueira et al. (2005).

In what follows, we present the process that the ELECTRE III method is utilized to tackle Example 4.3 (Chen et al. 2013c):

Step 1. Construction of fuzzy group decision matrix:
As mentioned previously, in the hesitant fuzzy decision matrix $H=\left(h_{i j}\right)_{4 \times 4}$, each element $h_{i j}=\left\{\gamma \mid \gamma \in h_{i j}, 0 \leq \gamma \leq 1\right\}$ indicates the possible membership degrees of the $i$ th alternative $A_{i}$ under the $j$ th attribute by the DMs.

To derive the group decision matrix, we aggregate the DMs' individual decision information with the averaging operator, which is defined by $\tilde{h}_{i j}=\frac{1}{l_{h_{i j}}} \sum_{\gamma \in h_{i j}} \gamma$. As it can be easily seen that the results obtained from the formula are just those of the score function mentioned earlier, which has been given in Table 4.5. So, the fuzzy group decision matrix $\tilde{H}=\left(\tilde{h}_{i j}\right)_{4 \times 4}$ is obtained.

Step 2. Indifference, preference and veto threshold values on an attribute $x_{j}$ are introduced by the DMs and given in Table 4.20 (Chen et al. 2013c).

Table 4.20. Three threshold values

| Threshold values | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $q_{j}$ | 0.02 | 0.05 | 0.02 | 0.02 |
| $p_{j}$ | 0.1 | 0.1 | 0.05 | 0.1 |
| $o_{j}$ | 0.3 | 0.3 | 0.2 | 0.3 |

Step 3. Calculate the partial concordance index $c_{j}\left(A_{i}, A_{k}\right)$ with Eq.(4.35) and summarize the results in Table 4.21 (Chen et al. 2013c).

Table 4.21. Partial concordance index for each attribute $x_{j}$

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $c_{1}\left(A_{i}, A_{k}\right)$ | $A_{1}$ | 1 | 0 | 0.1038 |
|  | $A_{2}$ | 1 | 1 | 0.8325 |
| $c_{2}\left(A_{i}, A_{k}\right)$ | $A_{3}$ | 1 | 0.9375 | 1 |
|  |  |  |  |  |
|  | $A_{4}$ | 1 | 0.2088 | 0.5213 |
| 1 |  |  |  |  |
|  | $A_{2}$ | 0 | 1 | 1 |
| 1 |  |  |  |  |
|  | $A_{3}$ | 0 | 1 | 0 |
| 1 |  |  |  |  |
|  | $A_{4}$ | 0 | 1 | 1 |

Table 4.21. (continued)

| $c_{3}\left(A_{i}, A_{k}\right)$ | $A_{1}$ | 1 | 1 | 0.5 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $A_{2}$ | 1 | 1 | 0.8333 | 0 |
|  | $A_{3}$ | 1 | 1 | 1 | 0.8333 |
| $c_{4}\left(A_{i}, A_{k}\right)$ | $A_{4}$ | 1 | 1 | 1 | 1 |
|  | $A_{2}$ | 1 | 1 | 1 | 0 |
|  | $A_{3}$ | 0.75 | 0 | 1 | 0 |
|  | $A_{4}$ | 1 | 1 | 1 | 1 |

Step 4. Calculate the overall concordance index $c\left(A_{i}, A_{k}\right)$ with Eq.(4.36) and summarize the results in Table 4.22 (Chen et al. 2013c).

Table 4.22. Concordance matrix

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 1 | 0.45 | 0.7458 | 0.4665 |
| $A_{2}$ | 0.7 | 1 | 0.6750 | 0.5 |
| $A_{3}$ | 0.6125 | 0.6375 | 1 | 0.6250 |
| $A_{4}$ | 0.7 | 0.8418 | 0.6043 | 1 |

Step 5. Calculate the discordance index $d_{j}\left(A_{i}, A_{k}\right)$ with Eq.(4.37) and summarize the results in Table 4.23 (Chen et al. 2013c).

Table 4.23. Discordance index for each attribute $x_{j}$

|  |  | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: | :---: |

Step 6. Calculate the credibility degree $\bar{S}\left(A_{i}, A_{k}\right)$ with Eq.(4.38) and summarize the results in Table 4.24 (Chen et al. 2013c).

Table 4.24. Credibility matrix

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 1 | 0.45 | 0.7458 | 0 |
| $A_{2}$ | 0.7 | 1 | 0.6750 | 0.5 |
| $A_{3}$ | 0.6125 | 0.6375 | 1 | 0 |
| $A_{4}$ | 0.7 | 0.8418 | 0.6043 | 1 |

Using the exploiting procedure from the credibility matrix, the result of outranking is $A_{4} \succ A_{2} \succ A_{3} \succ A_{1}$.

We next analyze the calculations with the ELECTRE IV method.
Let two alternatives $A_{1}$ and $A_{2}$ belong to a given set of actions, and $y_{j}\left(A_{1}\right)$ and $y_{j}\left(A_{2}\right)$ the performances of $A_{1}$ and $A_{2}$ in terms of the attribute $x_{j}$. We denote indifference, preference and veto thresholds on the attribute introduced by the DMs with $q_{j}, p_{j}$ and $o_{j}$, respectively, where $j=1,2, \cdots, n . J^{+}\left(A_{1}, A_{2}\right)$ and $J^{-}\left(A_{1}, A_{2}\right)$ respectively represent the sums of all those attributes where the performance of $A_{1}$ is superior and inferior to $A_{2}$, that is,

$$
\begin{gather*}
J^{+}\left(A_{1}, A_{2}\right)=\left\{j \in J \mid j: y_{j}\left(A_{2}\right)+q_{j}\left(y_{j}\left(A_{2}\right)\right)<y_{j}\left(A_{1}\right)\right\}  \tag{4.39}\\
J^{-}\left(A_{1}, A_{2}\right)=\left\{j \in J \mid j: y_{j}\left(A_{1}\right)+q_{j}\left(y_{j}\left(A_{1}\right)\right)<y_{j}\left(A_{2}\right)\right\} \tag{4.40}
\end{gather*}
$$

The ELECTRE IV method contains two levels of the outranking relations; The strong outranking relation $S^{+}$and the weak outranking relation $S^{-}$, which are defined by

$$
\begin{equation*}
A_{1} S^{+} A_{2} \Leftrightarrow \forall j, y_{j}\left(A_{1}\right)+p_{j}\left(y_{j}\left(A_{1}\right)\right) \geq y_{j}\left(A_{2}\right) \text { and }\left\|J^{+}\left(A_{1}, A_{2}\right)\right\|>\left\|J^{-}\left(A_{1}, A_{2}\right)\right\| \tag{4.41}
\end{equation*}
$$

$$
\begin{align*}
& A_{1} S^{-} A_{2} \Leftrightarrow \forall j, y_{j}\left(A_{1}\right)+p_{j}\left(y_{j}\left(A_{1}\right)\right) \geq y_{j}\left(A_{2}\right) \\
& \qquad \text { or }\left\{\begin{array}{l}
\exists k, y_{k}\left(A_{1}\right)+o_{k}\left(y_{k}\left(A_{1}\right)\right) \geq y_{k}\left(A_{2}\right)>y_{k}\left(A_{1}\right)+p_{k}\left(y_{k}\left(A_{1}\right)\right) \\
\text { and }\left\|j: y_{j}\left(A_{1}\right)-y_{j}\left(A_{2}\right)>p_{j}\left(y_{j}\left(A_{1}\right)\right)\right\| \geq \frac{n}{2}
\end{array}\right. \tag{4.42}
\end{align*}
$$

where $\|J\|$ denotes the number of elements in the set $J$.
The ELECTRE IV method exploiting procedure is the same as in the ELECTRE III method (Buchanan and Vanderpooten 2007; Figueira et al. 2005).

The details of dealing with Example 4.3 with the ELECTRE IV method are given (Chen et al. 2013c):

Step 1. The process is the same as that with the ELECTRE III method.

Step 2. Indifference, preference and veto threshold values on the attribute $x_{j}$ are the same as those given in Table 4.20.

Step 3. Calculate $J^{+}\left(A_{i}, A_{k}\right)$ and $J^{-}\left(A_{i}, A_{k}\right)$ with Eqs.(4.39) and (4.40). The results are summarized in Table 4.25.

Step 4. Calculate the outranking relation $S$ between alternatives with Eqs.(4.41) and (4.42). The results are also displayed in Table 4.25 (Chen et al. 2013c).

Table 4.25. $J^{+}\left(A_{i}, A_{k}\right), J^{-}\left(A_{i}, A_{k}\right)$ and the outrankings

|  |  | $A_{1}$ |  |  | $A_{2}$ |  |  | $A_{3}$ |  |  | $A_{4}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $J^{+}$ | $J^{-}$ | $S$ | $J^{+}$ | $J^{-}$ | $\bar{S}$ | $J^{+}$ | $J^{-}$ | $S$ | $J^{+}$ | $J^{-}$ | $\bar{S}$ |
| $A_{1}$ |  |  |  | \{2\} | \{1,4\} |  | \{2,4\} | \{1,3\} | S | \{2\} | \{1,3,4\} |  |
| $A_{2}$ | \{1,4\} | \{2\} | $s$ |  |  |  | \{1,4\} | \{2,3\} |  | \{1\} | \{3,4\} |  |
| $A_{3}$ | \{1,3\} | \{2,4\} |  | \{2,3\} | \{1,4\} |  |  |  |  | \{1,2\} | \{3,4\} |  |
| $A_{4}$ | \{1,3,4\} | \{2\} | $S$ | \{3,4\} | \{1\} | $S^{+}$ | \{3,4\} | \{1,2\} |  |  |  |  |

We can see from Table 4.25 that $A_{1} \succ_{S^{-}} A_{3}, A_{2} \succ_{S^{-}} A_{1}, A_{4} \succ_{S^{-}} A_{1}$, and $A_{4} \succ_{s^{+}} A_{2}$.

Step 5. Draw the strong and weak outranking graphs in Fig. 4.10 (Chen et al. 2013c).


With the exploiting procedure, the resulting outranking is $A_{4} \succ A_{2} \succ A_{1} \succ A_{3}$.
One can see the conclusions obtained with the ELECTRE III and ELECTRE IV methods are partially consistent with that derived with the HF-ELECTRE I methods, whereas a difference in the outranking relations is also noticed. We explain it in the following way: In the ELECTRE III and ELECTRE IV group outranking methods, because group decision information is obtained through averaging for all DMs, so only information involving the average opinion of all DMs is considered, whereas in the HF-ELECTRE I method, in addition to that consideration, the deviation degrees which reflect the difference in opinions between the individual DMs and their averages are also accounted for. It should be pointed out that the ELECTRE III and ELECTRE IV methods can give the ranking of all alternatives, which is different from the HF-ELECTRE I method that gives the partial outranking relations, thus, these two kinds of group decision making approaches are complementary.

### 4.3 Interactive Decision Making Method under Hesitant Fuzzy Environment with Incomplete Weight Information

### 4.3.1 Satisfaction Degree Based Models for MADM with Incomplete Weight Information

According to the interpretation of the special HFEs in Subsection 1.1.1, we know that the alternative whose ratings over attributes are all full sets is ideal and desirable, and similarly the alternative whose evaluation values over attributes are all empty sets is negative and undesirable. If all ratings of the alternative $A_{i}$ over
the attributes $x_{j}(j=1,2, \cdots, n)$ are full sets, then the overall value of $A_{i}$ can be calculated easily (also using the AHFWA operator (3.25) as an illustration):

$$
\begin{equation*}
g^{+}=A H F W A(\{1\},\{1\}, \cdots,\{1\})=\oplus_{j=1}^{n}\left(w_{j}\{1\}\right)=\left\{1-\prod_{j=1}^{n}(1-1)^{w_{j}} \mid t=1,2, \cdots, l\right\}=\{1\} \tag{4.43}
\end{equation*}
$$

Analogously, if all evaluation values of the alternative $A_{i}$ over the attributes $x_{j}(j=1,2, \cdots, n)$ are empty sets, then the overall value of $A_{i}$ can be calculated:

$$
\begin{equation*}
g^{-}=A H F W A(\{0\},\{0\}, \cdots,\{0\})=\oplus_{j=1}^{n}\left(w_{j}\{0\}\right)=\left\{1-\prod_{j=1}^{n}(1-0)^{w_{j}} \mid t=1,2, \cdots, l\right\}=\{0\} \tag{4.44}
\end{equation*}
$$

Definition 4.4. (Liao and Xu 2013a) $h^{+}=(\{1\},\{1\}, \cdots,\{1\})$ is called the hesitant fuzzy positive ideal solution, and $h^{-}=(\{0\},\{0\}, \cdots,\{0\})$ is called the hesitant fuzzy negative ideal solution.

For any alternative $A_{i}$ with the ratings $h_{i}=\left(h_{i 1}, h_{i 2}, \cdots, h_{i n}\right)$, based on the distance measure of HFEs mentioned in Section 2.1, we can calculate the distance between the alternative $A_{i}$ and the hesitant fuzzy positive ideal solution $h^{+}$and also the distance between the alternative $A_{i}$ and the hesitant fuzzy negative ideal solution $h^{-}$, respectively. Combining (3.25), (4.43), (4.44), and taking hesitant normalized hamming distance as an illustration, we can derive the distances according to their overall values:

$$
\begin{align*}
d_{\text {Hamming }}\left(g_{i}, g^{+}\right) & =\frac{1}{l} \sum_{t=1}^{l}\left|\left(1-\prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{w_{j}}\right)-1\right| \\
& =\frac{1}{l} \sum_{t=1}^{l} \prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{w_{j}}  \tag{4.45}\\
d_{\text {Hamming }}\left(g_{i}, g^{-}\right) & =\frac{1}{l} \sum_{t=1}^{l}\left|\left(1-\prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{w_{j}}\right)-0\right|
\end{align*}
$$

$$
\begin{equation*}
=\frac{1}{l} \sum_{t=1}^{l}\left(1-\prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{w_{j}}\right) \tag{4.46}
\end{equation*}
$$

where $l=\max \left\{l_{h_{i 1}}, l_{h_{i 2}}, \cdots, l_{h_{i n}}\right\}$.
Since $\quad g_{i}(w)=\left\{1-\prod_{j=1}^{n}\left(1-h_{i j}^{\sigma(t)}\right)^{w_{j}} \mid t=1,2, \cdots, l ; l=\max \left\{l_{h_{i 1}}, l_{h_{i 2}}, \cdots, l_{h_{i n}}\right\}\right\}$, then can see that $d_{\text {Hamming }}\left(g_{i}, g^{-}\right)=s\left(g_{i}\right)$, i.e., the distance between $g_{i}$ and $g^{-}$equals the score of $g_{i}$, which is just a coincidence.

Ostensibly, all of these derivations seem to be quite easy. However, we may ignore an importance precondition, in which the weight information is partially known and we can not get the crisp weights corresponding to different attributes. Consequently, it is hard or impossible to calculate the distance between the alternatives $A_{i}(i=1,2, \cdots, n)$ and the hesitant fuzzy positive ideal solution $h^{+}$ and the distance between the alternatives $A_{i}(i=1,2, \cdots, n)$ and the hesitant fuzzy negative ideal solution $h^{-}$by using Eqs.(4.45) and (4.46). Even to determinate the weights is also a difficult or insurmountable question due to the fact that the unknown parameters $w_{j}(j=1,2, \cdots, n)$ are in the exponential term. Hence, we need to find a novel way to solve this issue.

Reconsider the main idea of what we have done above. We firstly fuse the values $h_{i j} \quad(j=1,2, \cdots, m)$ of the alternative $A_{i}$ over the attributes $x_{j}(j=1,2, \cdots, m)$ by using some developed operators, and then calculate the distance between the derived overall values $g_{i}$ and $g^{+}$or $g^{-}$. What about changing the order of the process? If we firstly calculate the distance between each rating and the hesitant fuzzy positive ideal point or the hesitant fuzzy negative ideal point, and then fuse the distances with respect to the attributes $x_{j}(j=1,2, \cdots, n)$ according to some developed aggregation operators, the computational complexity will be changed significantly, which makes the problem easy to handle.

Since

$$
d_{\text {Hamming }}\left(h_{i j}, h_{i j}^{+}\right)=\frac{1}{l} \sum_{t=1}^{l}\left|h_{i j}^{\sigma(t)}-1\right|=\frac{1}{l} \sum_{t=1}^{l}\left(1-h_{i j}^{\sigma(t)}\right)
$$

$$
\begin{gather*}
=1-\frac{1}{l} \sum_{t=1}^{l} h_{i j}^{\sigma(t)}=1-s\left(h_{i j}\right)  \tag{4.47}\\
d_{\text {Hamming }}\left(h_{i j}, h_{i j}^{-}\right)=\frac{1}{l} \sum_{t=1}^{l}\left|h_{i j}^{\sigma(t)}-0\right|=\frac{1}{l} \sum_{t=1}^{l} h_{i j}^{\sigma(t)}=s\left(h_{i j}\right) \tag{4.48}
\end{gather*}
$$

then we can fuse the distances with respect to different attributes by some developed operators. These operators are not limited by the hesitant fuzzy aggregation operators but all of the classical operators ( Xu and Da 2003) because the distance $d_{\text {Hamming }}\left(h_{i j}, h_{i j}^{+}\right)$and $d_{\text {Hamming }}\left(h_{i j}, h_{i j}^{-}\right)$are all crisp values. Taking the weighted averaging operator as an example, the overall distance between the alternative $A_{i}$ and the hesitant fuzzy positive ideal solution $h^{+}$and also the distance between the alternative $A_{i}$ and the hesitant fuzzy negative ideal solution $h^{-}$can be derived respectively:

$$
\begin{gather*}
d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{+}\right)=\sum_{j=1}^{m} w_{j}\left(1-s\left(h_{i j}\right)\right)=1-\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)  \tag{4.49}\\
d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{-}\right)=\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right) \tag{4.50}
\end{gather*}
$$

Intuitively, the smaller the distance $d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{+}\right)$, the better the alternative; While the larger the distance $d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{-}\right)$, the better the alternative. Motivated by the TOPSIS method (Hwang and Yoon 1981; Chen and Hwang 1992), we shall take both of the distance $d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{+}\right)$and the distance $d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{-}\right)$into consideration simultaneously rather than consider them alone. Then we derive the definition of satisfaction degree naturally.

Definition 4.5 (Liao and Xu 2013a). A satisfaction degree of the given alternative $A_{i}$ over the attributes $x_{j}(j=1,2, \cdots, m)$ with the weight vector $w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}} \in \Delta$ (here $\Delta$ is a set of the known weight information shown in Section 4.1) is defined as:

$$
\begin{equation*}
\bar{\rho}\left(g_{i}(w)\right)=\frac{d_{\text {Hamming }}\left(g_{i}, g^{-}\right)}{d_{\text {Hamming }}\left(g_{i}, g^{+}\right)+d_{\text {Hamming }}\left(g_{i}, g^{-}\right)} \tag{4.51}
\end{equation*}
$$

where $0 \leq w_{j} \leq 1, \quad j=1,2, \cdots, m$, and $\sum_{j=1}^{m} w_{j}=1$.
Combining Eqs.(4.49), (4.50) and (4.51), we have

$$
\begin{align*}
\bar{\rho}\left(g_{i}(w)\right) & =\frac{d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{-}\right)}{d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{+}\right)+d_{\text {Hamming }}^{\prime}\left(g_{i}, g^{-}\right)} \\
& =\frac{\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)}{1-\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)+\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)}=\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right) \tag{4.52}
\end{align*}
$$

From Eqs.(4.50) and (4.52), we can see that the satisfaction degree reduces to the distance between the alternative $A_{i}$ and the hesitant fuzzy negative ideal solution $h^{-}$, which is a coincidence and the property of HFSs. In order not to lose quite much information and make our method more applicable, we introduce a parameter $\theta$, which denotes the risk preferences of the decision makers: $\theta>0.5$ means the DMs are pessimists and the further the distance between the alternative and the positive ideal solution, the better the choice; While $\theta<0.5$ means the opposite. Consequently, the satisfaction degree becomes

$$
\begin{equation*}
\bar{\rho}\left(g_{i}(w)\right)=\frac{(1-\theta) \sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)}{\theta\left(1-\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)\right)+(1-\theta) \sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)} \tag{4.53}
\end{equation*}
$$

The value of the parameter $\theta$ is provided by the DM in advance. It is obvious that $0 \leq \bar{\rho}\left(g_{i}(w)\right) \leq 1$, for any $\theta \in[0,1], i=1,2, \cdots, n$. As our purpose is to select the alternative with the highest satisfaction degree, the following
multi-objective optimization model can be generated naturally (Liao and Xu 2013a):
(M-4.1)

$$
\max \left(\bar{\rho}\left(g_{1}(w)\right), \bar{\rho}\left(g_{2}(w)\right), \cdots, \bar{\rho}\left(g_{n}(w)\right)\right)
$$

$$
\begin{array}{ll}
\text { s.t. } & w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}} \in \Delta \\
& 0 \leq w_{j} \leq 1, \quad j=1,2, \cdots, m \\
\sum_{j=1}^{m} w_{j}=1
\end{array}
$$

We change the model (M-4.1) into a single-objective optimization model by using the equal weighted summation method (French et al. 1983):
(M-4.2)

$$
\begin{gathered}
\max \sum_{i=1}^{n} \bar{\rho}\left(g_{i}(w)\right) \\
\text { s.t. } w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}} \in \Delta \\
0 \leq w_{j} \leq 1, \quad j=1,2, \cdots, m \\
\sum_{j=1}^{m} w_{j}=1 .
\end{gathered}
$$

Combining Eq.(4.53) and the model (M-4.2), Liao and Xu (2013a) established the following model:
(M-4.3)

$$
\begin{aligned}
& \max \sum_{i=1}^{n} \frac{(1-\theta) \sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)}{\theta\left(1-\sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)\right)+(1-\theta) \sum_{j=1}^{m} w_{j} s\left(h_{i j}\right)} \\
& \text { s.t. } \quad w=\left(w_{1}, w_{2}, \cdots, w_{n}\right)^{\mathrm{T}} \in \Delta, \\
& \quad 0 \leq w_{j} \leq 1, \quad j=1,2, \cdots, m,
\end{aligned}
$$

$$
\sum_{j=1}^{m} w_{j}=1 .
$$

which can be solved by using many efficient algorithms (Terlaky 1996) or using the MATLAB or the Lingo mathematic software package. Suppose that the optimal solution of the model (M-4.3) is $w^{*}=\left(w_{1}^{*}, w_{2}^{*}, \cdots, w_{n}^{*}\right)^{\mathrm{T}}$, then we can calculate the overall value $g_{i}$ of each alternative $A_{i}$ according to Eq.(3.25). Subsequently, the ranking order of alternatives can be derived by using the comparison method introduced previously.

We now consider a decision making problem that concerns the evaluation of the service quality among domestic airlines (Liou et al. 2011) to illustrate the model (M-4.3):

Example 4.4 (Liao and Xu 2013a). Due to the development of high-speed railroad, the domestic airline marketing has faced a stronger challenge in Taiwan. Especially after the global economic downturn in 2008, more and more airlines have attempted to attract customers by reducing price. Unfortunately, they soon found that this is a no-win situation and only service quality is the critical and fundamental element to survive in this highly competitive domestic market. In order to improve the service quality of domestic airline, the civil aviation administration of Taiwan (CAAT) wants to know which airline is the best in Taiwan. So the CAAT constructs a committee to investigate the four major domestic airlines, which are UNI Air, Transasia, Mandarin, and Daily Air and four major criteria are given to evaluate these four domestic airlines. These four main attributes are:
(1) $x_{1}$ : Booking and ticketing service, which involves convenience of booking or buying ticket, promptness of booking or buying ticket, courtesy of booking or buying ticket;
(2) $x_{2}$ : Check-in and boarding process, which consists of convenience check-in, efficient check-in, courtesy of employee, clarity of announcement and so on;
(3) $x_{3}$ : Cabin service, which can be divided into cabin safety demonstration, variety of newspapers and magazines, courtesy of flight attendants, flight attendant willing to help, clean and comfortable interior, in-flight facilities, and captain's announcement;
(4) $x_{4}$ : Responsiveness, which consists of fair waiting-list call, handing of delayed flight, complaint handing, and missing baggage handling.

There is no doubt that finding the best practice in each of the four main attributes and then calling all companies to learn from them respectively is better than determining the best company as a whole and trying to make the others follow all its practices, due to the fact that some of them would be inferior to the practice of some of the "followers". However, selecting the best airline as a whole over all attributes
is also very important especially for the passengers. Meanwhile, it is also very useful for the company to achieve brand effect.

Suppose that the committee gives the rating values by using HFEs, and then the hesitant fuzzy decision matrix $H$ is presented in Table 4.26 (Liao and Xu 2013a):

Table 4.26. Hesitant fuzzy decision matrix

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| UNI Air | $\{0.6,0.7,0.9\}$ | $\{0.6,0.8\}$ | $\{0.3,0.6,0.9\}$ | $\{0.4,0.5,0.9\}$ |
| Transasia | $\{0.7,0.8,0.9\}\{0.5,0.8,0.9\}$ | $\{0.4,0.8\}$ | $\{0.5,0.6,0.7\}$ |  |
| Mandarin | $\{0.5,0.6,0.8\}$ | $\{0.6,0.7,0.9\}$ | $\{0.3,0.5,0.7\}$ | $\{0.5,0.7\}$ |
| Daily Air | $\{0.6,0.9\}$ | $\{0.7,0.9\}$ | $\{0.2,0.4,0.7\}$ | $\{0.4,0.5\}$ |

The weight information of these four attributes is given like this:

$$
\begin{aligned}
& \Delta=\left\{w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}} \mid w_{3} \geq w_{4} \geq w_{2} \geq w_{1} \geq 0.1,0.5 \geq w_{3} \geq 0.4\right. \\
& \left.w_{2} \leq 2 w_{1}, \sum_{j=1}^{4} w_{j}=1\right\}
\end{aligned}
$$

Firstly, we calculate the distance between each rating and the hesitant fuzzy positive ideal point or the hesitant fuzzy negative ideal point by using Eqs.(4.45) and (4.46). To do so, we only need to compute the score matrix $S$ :

$$
S=\left(\begin{array}{cccc}
0.7333 & 0.7 & 0.6 & 0.6 \\
0.8 & 0.7333 & 0.6 & 0.6 \\
0.6333 & 0.7333 & 0.5 & 0.6 \\
0.75 & 0.8 & 0.4333 & 0.45
\end{array}\right)
$$

In this example, we take $\theta=0.4$ as an illustration. Based on the score function value matrix $S$ and the partially known weight information $\Delta$, the model (M-4.3) can be established as follows:

## (M-4.4)

$$
\begin{aligned}
& \max \frac{0.6\left(0.7333 w_{1}+0.7 w_{2}+0.6 w_{3}+0.6 w_{4}\right)}{0.4\left(0.2667 w_{1}+0.3 w_{2}+0.4 w_{3}+0.4 w_{4}\right)+0.6\left(0.7333 w_{1}+0.7 w_{2}+0.6 w_{3}+0.6 w_{4}\right)} \\
& +\frac{0.6\left(0.8 w_{1}+0.7333 w_{2}+0.6 w_{3}+0.6 w_{4}\right)}{0.4\left(0.2 w_{1}+0.2667 w_{2}+0.4 w_{3}+0.4 w_{4}\right)+0.6\left(0.8 w_{1}+0.7333 w_{2}+0.6 w_{3}+0.6 w_{4}\right)} \\
& +\frac{0.6\left(0.6333 w_{1}+0.7333 w_{2}+0.5 w_{3}+0.6 w_{4}\right)}{0.4\left(0.3667 w_{1}+0.2667 w_{2}+0.5 w_{3}+0.4 w_{4}\right)+0.6\left(0.6333 w_{1}+0.7333 w_{2}+0.5 w_{3}+0.6 w_{4}\right)} \\
& +\frac{0.6\left(0.75 w_{1}+0.8 w_{2}+0.4333 w_{3}+0.45 w_{4}\right)}{0.4\left(0.25 w_{1}+0.2 w_{2}+0.5667 w_{3}+0.55 w_{4}\right)+0.6\left(0.75 w_{1}+0.8 w_{2}+0.4333 w_{3}+0.45 w_{4}\right)}
\end{aligned}
$$

$$
\text { s.t. } w_{3} \geq w_{4} \geq w_{2} \geq w_{1} \geq 0.1
$$

$$
0.5 \geq w_{3} \geq 0.4 ; w_{2} \leq 2 w_{1}
$$

$$
\sum_{j=1}^{4} w_{j}=1
$$

The objective function can be simplified as:

$$
\begin{aligned}
& \max \frac{0.44 w_{1}+0.42 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.5467 w_{1}+0.54 w_{2}+0.52 w_{3}+0.52 w_{4}}+\frac{0.48 w_{1}+0.44 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.56 w_{1}+0.5467 w_{2}+0.52 w_{3}+0.52 w_{4}} \\
& +\frac{0.38 w_{1}+0.44 w_{2}+0.3 w_{3}+0.36 w_{4}}{0.5267 w_{1}+0.5467 w_{2}+0.5 w_{3}+0.2 w_{4}}+\frac{0.45 w_{1}+0.48 w_{2}+0.26 w_{3}+0.27 w_{4}}{0.55 w_{1}+0.56 w_{2}+0.4867 w_{3}+0.49 w_{4}}
\end{aligned}
$$

By solving this model, we get the optimal solution:

$$
w^{*}=\left(w_{1}^{*}, w_{2}^{*}, w_{3}^{*}, w_{4}^{*}\right)^{\mathrm{T}}=(0.12,0.24,0.4,0.24)^{\mathrm{T}}
$$

According to Eq.(3.25), the overall value of each domestic airline can be obtained as follows (the extended hesitant fuzzy matrix whose elements are with equal length can be seen in Table 4.27 (Liao and Xu 2013a)):

$$
\begin{aligned}
& g_{1}\left(w^{*}\right)=\{0.4485,0.5293,0.8819\} \\
& g_{2}\left(w^{*}\right)=\{0.4942,0.6335,0.8282\} \\
& g_{3}\left(w^{*}\right)=\{0.4578,0.5694,0.7805\} \\
& g_{4}\left(w^{*}\right)=\{0.4571,0.5161,0.7716\}
\end{aligned}
$$

Table 4.27. Extended hesitant fuzzy decision matrix
$\left.\begin{array}{cccc}\hline & x_{1} & x_{2} & x_{3}\end{array}\right] x_{4}$,

Hence, the scores of the overall values $g_{i}\left(w^{*}\right)(i=1,2,3,4)$ are

$$
\begin{aligned}
& s\left(g_{1}\left(w^{*}\right)\right)=0.6199, s\left(g_{2}\left(w^{*}\right)\right)=0.6520 \\
& s\left(g_{3}\left(w^{*}\right)\right)=0.6026, s\left(g_{4}\left(w^{*}\right)\right)=0.5816
\end{aligned}
$$

respectively.
Since $s\left(g_{2}\left(w^{*}\right)\right)>s\left(g_{1}\left(w^{*}\right)\right)>s\left(g_{3}\left(w^{*}\right)\right)>s\left(g_{4}\left(w^{*}\right)\right)$, then we can rank these four domestic airlines in descending order as $g_{2}\left(w^{*}\right)>g_{1}\left(w^{*}\right)>g_{3}\left(w^{*}\right)>g_{4}\left(w^{*}\right)$.

That is to say, the service quality of Transasia is the best among the service quality of domestic airline in Taiwan.

Now let's reconsider the method again deeply. In general, in the process of decision making with incomplete weight information on the attributes, the basic thing we should do is to find the weights that are as adequate as possible to the opinions of the DMs. Does the given model reflect the DMs' opinions? The answer is "yes". Actually, starting from calculating the overall distance between the alternative $A_{i}$ and the hesitant fuzzy positive ideal solution $h^{+}$and the distance between the alternative $A_{i}$ and the hesitant fuzzy negative ideal solution $h^{-}$by using Eqs.(4.47) and (4.48), respectively, the DMs' opinions have been taken into account. The aim of introducing the satisfaction degree is also to model the DMs' opinions more comprehensive since it includes both of the above two distances. In addition, the parameter $\theta$, which denotes the risk preferences of the DMs , is also used to enhance the reflection of the DMs’ ideas. Since the unknown weight information can not be obtained directly, maximizing those satisfaction degrees simultaneously is a good choice to find a solution which does not show any discrimination to certain alternative(s), and meanwhile reflects the DMs' opinions comprehensively. Certainly we can also minimize those satisfaction degrees simultaneously if we want to select the worst alternative(s).

### 4.3.2 Interactive Method for MADM under Hesitant Fuzzy Environment with Incomplete Weight Information

In the previous subsection, we have presented the satisfaction degree based models to handle the MADM problem whose weight information is partially known. However, by using these models, the satisfaction degrees of certain alternatives are sometimes too high and simultaneously others are too low. Satisfaction degrees with a wide range may match with some DMs' requirement, but, very often, in the process of decision making, the DMs may want to modify their satisfaction degrees slightly in order to provide new preference information or modify the previous preference information. Interacting with the DMs gradually is an acceptable and applicable way for doing so in reality. Interactive decision making as a hot topic has been studied by many scholars recently, in the situations where the ratings are given in HFEs, Liao and Xu (2013a) proposed an interactive method for MADM with hesitant fuzzy information:

The main idea of this method can be clarified like this: Firstly, the DMs give the lower bounds of the satisfaction degrees with respect to each alternative, and then according to these lower bounds, we can establish the weights of different attributes. Once we have determined the different weights, the satisfaction degrees of different alternatives can be calculated easily and the analysts then ask the DMs whether they want to reconsider the satisfaction degrees or not. If the DMs are not satisfied with the derived satisfaction degrees, then the analysts shall inform the

DMs to reconsider their lower bounds of the satisfaction degrees and then go to do iteration till acceptable.

In the following, we introduce this method in details. In order to help the DMs establish the lower bounds of the alternatives, motivated by the max-min operator developed by Zimmermann and Zysno (1980), we derive the following model (Liao and Xu 2013a):
(M-4.5)
$\max \lambda$

$$
\begin{aligned}
& \text { s.t. } \quad \bar{\rho}\left(g_{i}(w)\right) \geq \lambda, \\
& w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}} \in \Delta, \\
& 0 \leq w_{j} \leq 1, j=1,2, \cdots, m, \sum_{j=1}^{m} w_{j}=1 .
\end{aligned}
$$

By solving the model (M-4.5), we obtain the initial optimal weight vector $w^{(0)}=\left(w_{1}^{(0)}, w_{2}^{(0)}, \cdots, w_{n}^{(0)}\right)^{\mathrm{T}} \quad$ and the initial satisfaction degrees $\bar{\rho}\left(g_{i}\left(w^{(0)}\right)\right)(i=1,2, \cdots, n)$ of the alternatives $A_{i}(i=1,2, \cdots, n)$. In the process of MADM, the DMs can provide the lower bounds $\lambda_{i}^{(0)}(i=1,2, \cdots, n)$ of the satisfaction degrees of the alternatives $A_{i}(i=1,2, \cdots, n)$ according to $\bar{\rho}\left(g_{i}\left(w^{(0)}\right)\right)(i=1,2, \cdots, n)$. Once we have got the lower bounds, the attribute weights can be reestablished by this model (Liao and Xu 2013a):
(M-4.6)

$$
\begin{aligned}
& \max \sum_{i=1}^{m} \lambda_{i} \\
& \text { s.t. } \bar{\rho}\left(g_{i}(w)\right) \geq \lambda_{i} \geq \lambda_{i}^{(0)}, i=1,2, \cdots, n \\
& w=\left(w_{1}, w_{2}, \cdots, w_{m}\right)^{\mathrm{T}} \in \Delta \\
& 0 \leq w_{j} \leq 1, j=1,2, \cdots, m, \sum_{j=1}^{m} w_{j}=1
\end{aligned}
$$

Solving this model we can get a new weight vector $w^{(1)}=\left(w_{1}^{(1)}, w_{2}^{(1)}, \cdots, w_{n}^{(1)}\right)^{\mathrm{T}}$. If this model has no optimal solution, this means
that some lower bounds are greater than the corresponding initial satisfaction degrees. Hence it needs to be reconsidered till the optimal solution is obtained.

For the convenience of application, the procedure of the interactive method for MADM under hesitant fuzzy environment with incomplete weights can be described as follows (Liao and Xu 2013a):

## (Algorithm 4.1)

Step 1. Construct the hesitant fuzzy decision matrix.
Step 2. Using the model (M-4.5) to determinate the initial weight vector $w^{(0)}=\left(w_{1}^{(0)}, w_{2}^{(0)}, \cdots, w_{m}^{(0)}\right)^{\mathrm{T}}$ and the initial satisfaction degrees $\bar{\rho}\left(g_{i}\left(w^{(0)}\right)\right)$ $(i=1,2, \cdots, n)$ of the alternatives $A_{i}(i=1,2, \cdots, n)$.

Step 3. The DMs can provide the lower bounds $\lambda_{i}^{(t)}(i=1,2, \cdots, m)$ of the satisfaction degrees of the alternatives $A_{i}(i=1,2, \cdots, n)$ according to $\bar{\rho}\left(g_{i}\left(w^{(0)}\right)\right)(i=1,2, \cdots, n)$. Let $t=t+1$.

Step 4. Solve the model (M-4.6) to determinate the weight vector $w^{(t)}=\left(w_{1}^{(t)}, w_{2}^{(t)}, \cdots, w_{m}^{(t)}\right)^{\mathrm{T}}$ and the satisfaction degrees $\bar{\rho}\left(g_{i}\left(w^{(t)}\right)\right)(i=1,2, \cdots, m)$ of the alternatives $A_{i}(i=1,2, \cdots, n)$.

Step 5. If the model has an optimal solution, then go to Step 6; Otherwise go to Step 3.

Step 6. Calculate the overall values $g_{i}\left(w^{(t)}\right)(i=1,2, \cdots, n)$ of the alternatives $A_{i}(i=1,2, \cdots, n)$ and rank them according to the comparison law, and then choose the best alternative.

Step 7. End.

Example 4.5 (Liao and Xu 2013a). To show the application of our algorithm, let's use Example 4.4 as an illustration again. The computational procedure can be set out as follows:

We also take $\theta=0.4$ as an illustration. Combining Example 4.4 and the model (M-4.5), the following optimal programming model can be derived easily:
(M-4.7) $\quad \max \lambda$

$$
\begin{gathered}
\text { s.t. } \frac{0.44 w_{1}+0.42 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.5467 w_{1}+0.54 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda, \\
\frac{0.48 w_{1}+0.44 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.56 w_{1}+0.5467 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda, \\
\frac{0.38 w_{1}+0.44 w_{2}+0.3 w_{3}+0.36 w_{4}}{0.5267 w_{1}+0.5467 w_{2}+0.5 w_{3}+0.2 w_{4}} \geq \lambda, \\
\frac{0.45 w_{1}+0.48 w_{2}+0.26 w_{3}+0.27 w_{4}}{0.55 w_{1}+0.56 w_{2}+0.4867 w_{3}+0.49 w_{4}} \geq \lambda \\
w_{3} \geq w_{4} \geq w_{2} \geq w_{1} \geq 0.1 \\
0.5 \geq w_{3} \geq 0.4 ; w_{2} \leq 2 w_{1} \\
\sum_{j=1}^{4} w_{j}=1
\end{gathered}
$$

By solving this model, we get the initial weight vector

$$
w^{(0)}=\left(w_{1}^{(0)}, w_{2}^{(0)}, \cdots, w_{n}^{(0)}\right)^{\mathrm{T}}=(0.2,0.2,0.4,0.2)^{\mathrm{T}}
$$

and the vector of the initial satisfaction degrees:
$\bar{\rho}^{(0)}=\left(\bar{\rho}^{(0)}\left(g_{1}\right), \bar{\rho}^{(0)}\left(g_{2}\right), \bar{\rho}^{(0)}\left(g_{3}\right), \bar{\rho}^{(0)}\left(g_{4}\right)\right)^{\mathrm{T}}=(73.3 \%, 75 \%, 78.3 \%, 66.84 \%)^{\mathrm{T}}$
of these four domestic airlines.

Suppose that the DMs provide the lower bounds of the satisfaction degrees of these four domestic airlines as:

$$
\lambda^{(0)}=\left(\lambda_{1}^{(0)}, \lambda_{2}^{(0)}, \lambda_{3}^{(0)}, \lambda_{4}^{(0)}\right)^{\mathrm{T}}=(70 \%, 75 \%, 80 \%, 70 \%)^{\mathrm{T}}
$$

and the model (M-4.6) becomes the following optimal programming problem:

$$
\begin{gathered}
\text { (M-4.8) } \max \left(\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}\right) \\
\text { s.t. } \frac{0.44 w_{1}+0.42 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.5467 w_{1}+0.54 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda_{1} \geq 0.7 \\
\frac{0.48 w_{1}+0.44 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.56 w_{1}+0.5467 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda_{2} \geq 0.75 \\
\frac{0.38 w_{1}+0.44 w_{2}+0.3 w_{3}+0.36 w_{4}}{0.5267 w_{1}+0.5467 w_{2}+0.5 w_{3}+0.2 w_{4}} \geq \lambda_{3} \geq 0.8 \\
\frac{0.45 w_{1}+0.48 w_{2}+0.26 w_{3}+0.27 w_{4}}{0.55 w_{1}+0.56 w_{2}+0.4867 w_{3}+0.49 w_{4}} \geq \lambda_{4} \geq 0.7 \\
\\
w_{3} \geq w_{4} \geq w_{2} \geq w_{1} \geq 0.1 \\
0.5 \geq w_{3} \geq 0.4 ; w_{2} \leq 2 w_{1} \\
\sum_{j=1}^{4} w_{j}=1
\end{gathered}
$$

Solving the model (M-4.8), we find there is no feasible solution. This may result from the fact that some of the lower bounds of the satisfaction degrees given by the DMs are too high. We appeal to the DMs and they then modify some of their lower bounds of the satisfaction degrees referring to the initial satisfaction degrees. Suppose that the modified lower bounds are

$$
\lambda^{(0)}=\left(\lambda_{1}^{(0)}, \lambda_{2}^{(0)}, \lambda_{3}^{(0)}, \lambda_{4}^{(0)}\right)^{\mathrm{T}}=(70 \%, 72 \%, 80 \%, 65 \%)^{\mathrm{T}}
$$

then we modify the model (M-4.8) into the model (M-4.9) (Liao and Xu 2013a):
(M-4.9)

$$
\begin{gathered}
\max \left(\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}\right) \\
\text { s.t. } \frac{0.44 w_{1}+0.42 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.5467 w_{1}+0.54 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda_{1} \geq 0.7 \\
\frac{0.48 w_{1}+0.44 w_{2}+0.36 w_{3}+0.36 w_{4}}{0.56 w_{1}+0.5467 w_{2}+0.52 w_{3}+0.52 w_{4}} \geq \lambda_{2} \geq 0.72 \\
\frac{0.38 w_{1}+0.44 w_{2}+0.3 w_{3}+0.36 w_{4}}{0.5267 w_{1}+0.5467 w_{2}+0.5 w_{3}+0.2 w_{4}} \geq \lambda_{3} \geq 0.8 \\
\frac{0.45 w_{1}+0.48 w_{2}+0.26 w_{3}+0.27 w_{4}}{0.55 w_{1}+0.56 w_{2}+0.4867 w_{3}+0.49 w_{4}} \geq \lambda_{4} \geq 0.65 \\
w_{3} \geq w_{4} \geq w_{2} \geq w_{1} \geq 0.1, \\
0.5 \geq w_{3} \geq 0.4 ; w_{2} \leq 2 w_{1} ; \sum_{j=1}^{4} w_{j}=1 .
\end{gathered}
$$

Solving this model, we get

$$
w^{(1)}=\left(w_{1}^{(1)}, w_{2}^{(1)}, w_{3}^{(1)}, w_{4}^{(1)}\right)^{\mathrm{T}}=(0.12,0.24,0.4,0.24)^{\mathrm{T}}
$$

and the satisfaction degrees:

$$
\bar{\rho}^{(1)}=\left(\bar{\rho}^{(1)}\left(g_{1}\right), \bar{\rho}^{(1)}\left(g_{2}\right), \bar{\rho}^{(1)}\left(g_{3}\right), \bar{\rho}^{(1)}\left(g_{4}\right)\right)^{\mathrm{T}}=(72.73 \%, 74.1 \%, 80.83 \%, 65.93 \%)^{\mathrm{T}}
$$

of these four domestic airlines. If the DMs are not satisfied with these results, they can further modify the lower bounds of the satisfaction degrees. Suppose that the DMs are satisfied with these results, then we go to the next step.

We further calculate the overall values $g_{i}\left(w^{(1)}\right)(i=1,2,3,4)$ of these four domestic airlines and rank them according to the comparison law. In analogy to Example 4.4, we can obtain $g_{2}\left(w^{(1)}\right)>g_{1}\left(w^{(1)}\right)>g_{3}\left(w^{(1)}\right)>g_{4}\left(w^{(1)}\right)$, and thus, the service quality of Transasia is best among the service quality of domestic airline in Taiwan. But from the satisfaction degrees, we can see
$\bar{\rho}_{3}\left(w^{(1)}\right) \succ \bar{\rho}_{2}\left(w^{(1)}\right) \succ \bar{\rho}_{1}\left(w^{(1)}\right) \succ \bar{\rho}_{4}\left(w^{(1)}\right)$. The reason for this is that when deriving the satisfaction degree, in order to decrease the computational complexity, we use Eqs.(4.47) and (4.48) to substitute Eqs.(4.43) and (4.44), respectively. So, after we have obtained the weights of the attributes, in order not to lose much information, we shall calculate the satisfaction degree by using Eq.(4.49) instead of Eq.(4.50). Hence, the service quality of Transasia is the best among the service quality of domestic airline in Taiwan.

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